10 Complex Variables

10.1 INTRODUCTION

In the course of developing tools for the solution of the variety of problems encountered in the physical sciences, we have had many occasions to use results from the theory of functions of a complex variable. The solution of a differential equation describing the motion of a spring–mass system is an example that comes immediately to mind. Functions of a complex variable also play an important role in fluid mechanics, heat transfer, and field theory, to mention just a few areas.

In this chapter we present a survey of this important subject. We progress from the complex numbers to complex variables to analytic functions to line integrals and finally to the famous residue theorem of Cauchy. This survey is not meant to be a treatise; the reader should consult any of the standard texts on the subject for a complete development.

10.1.1 *Maple* Applications

Maple commands for this chapter include: `Re`, `Im`, `argument`, `conjugate`, `evalc`, `unassign`, `taylor`, `laurent` (in the `numapprox` package), `convert/parfrac`, and `collect`, along with the commands from Appendix C. The symbol `I` is reserved for $\sqrt{-1}$.

10.2 COMPLEX NUMBERS

There are many algebraic equations such as

$$z^2 - 12z + 52 = 0 \tag{10.2.1}$$

which have no solutions among the set of real numbers. We have two alternatives; either admit that there are equations with no solutions, or enlarge the set of numbers so that every algebraic equation has a solution. A great deal of experience has led us to accept the second choice. We write $i = \sqrt{-1}$ so that $i^2 = -1$ and attempt to find solutions of the form

$$z = x + iy \tag{10.2.2}$$

where x and y are real. Equation 10.2.1 has solutions in this form:

$$\begin{aligned} z_1 &= 6 + 4i \\ z_2 &= 6 - 4i \end{aligned} \tag{10.2.3}$$

M. C. Potter et al., *Advanced Engineering Mathematics*,
https://doi.org/10.1007/978-3-030-17068-4_10

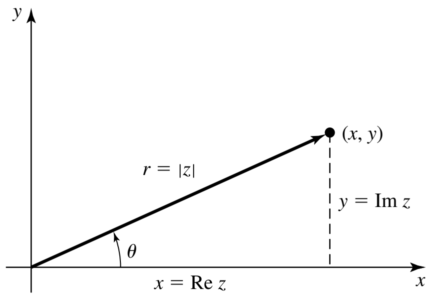


Figure 10.1 The complex plane.

For each pair of real numbers x and y, z is a *complex number*. One of the outstanding mathematical achievements is the theorem of Gauss, which asserts that every equation

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0 \tag{10.2.4}$$

has a solution in this enlarged set.[1] The complex number $z = x + iy$ has x as its *real part* and y as its *imaginary part* (y is real). We write

$$\operatorname{Re} z = x, \qquad \operatorname{Im} z = y \tag{10.2.5}$$

The notation $z = x + iy$ suggests a geometric interpretation for z. The point (x, y) is the plot of $z = x + iy$. Therefore, to every point in the xy plane there is associated a complex number and to every complex number a point. The x axis is called the *real axis* and the y axis is the *imaginary axis,* as displayed in Fig. 10.1.

In terms of polar coordinates (r, θ) the variables x and y are

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{10.2.6}$$

The complex variable z is then written as

$$z = r(\cos\theta + i\sin\theta) \tag{10.2.7}$$

The quantity r is the *absolute value* of z and is denoted by $|z|$; hence,

$$r = |z| = \sqrt{x^2 + y^2} \tag{10.2.8}$$

The angle θ, measured in radians and positive in the counterclockwise sense, is the *argument* of z, written $\arg z$ and given by

$$\arg z = \theta = \tan^{-1}\frac{y}{x} \tag{10.2.9}$$

Obviously, there are an infinite number of θ's satisfying Eq. 10.2.9 at intervals of 2π radians. We shall make the usual choice of limiting θ to the interval $0 \le \theta < 2\pi$ for its *principal value*[2]

[1]From this point, it is elementary to prove that this equation actually has n solutions, counting possible duplicates.

[2]Another commonly used interval is $-\pi < \theta \le \pi$.

A complex number is *pure imaginary* if the real part is zero. It is *real* if the imaginary part is zero. The *conjugate* of the complex number z is denoted by $\bar{z}$; it is found by changing the sign on the imaginary part of z, that is,

$$\bar{z} = x - iy \tag{10.2.10}$$

The conjugate is useful in manipulations involving complex numbers. An interesting observation and often useful result is that the product of a complex number and its conjugate is real. This follows from

$$z\bar{z} = (x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2 \tag{10.2.11}$$

where we have used $i^2 = -1$. Note then that

$$z\bar{z} = |z|^2 \tag{10.2.12}$$

The addition, subtraction, multiplication, or division of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is accomplished as follows:

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned} \tag{10.2.13}$$

$$\begin{aligned} z_1 - z_2 &= (x_1 - iy_1) - (x_2 + iy_2) \\ &= (x_1 - x_2) + i(y_1 - y_2) \end{aligned} \tag{10.2.14}$$

$$\begin{aligned} z_1z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned} \tag{10.2.15}$$

$$\begin{aligned} \frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} &= \frac{x_1 + iy_1}{x_2 + iy_2}\frac{x_2 - iy_2}{x_2 - iy_2} \\ &= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2} \end{aligned} \tag{10.2.16}$$

Note that the conjugate of z_2 was used to form a real number in the denominator of z_1/z_2. This last computation can also be written

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{z_1\bar{z}_2}{|z_2|^2} \tag{10.2.17}$$

$$= \frac{x_1x_2 + y_1y_2}{|z_2|^2} + i\frac{x_2y_1 - x_1y_2}{|z_2|^2} \tag{10.2.18}$$

Figure 10.1 shows clearly that

$$\begin{aligned} |x| &= |\mathrm{Re}\ z| \le |z| \\ |y| &= |\mathrm{Im}\ z| \le |z| \end{aligned} \tag{10.2.19}$$

Addition and subtraction is illustrated graphically in Fig. 10.2. From the parallelogram formed by the addition of the two complex numbers, we observe that

$$|z_1 + z_2| \le |z_1| + |z_2| \tag{10.2.20}$$

This inequality will be quite useful in later considerations.

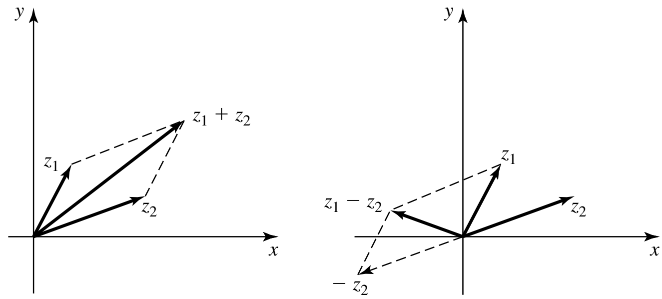


Figure 10.2 Addition and subtraction of two complex numbers.

We note the rather obvious fact that if two complex numbers are equal, the real parts and the imaginary parts are equal, respectively. Hence, an equation written in terms of complex variables includes two real equations, one found by equating the real parts from each side of the equation and the other found by equating the imaginary parts. Thus, for instance, the equation

$$a + ib = 0 \tag{10.2.21}$$

implies that $a = b = 0$.

When Eqs. 10.2.6 are used to write z as in Eq. 10.2.7, we say that z is in *polar* form. This form is particularly useful in computing $z_1 z_2$, z_1/z_2, z^n, and $z^{1/n}$. Suppose that

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1), \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2) \tag{10.2.22}$$

so that

$$\begin{aligned} z_1 z_2 &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\ &\qquad + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned} \tag{10.2.23}$$

It then follows that

$$|z_1 z_2| = r_1 r_2 = |z_1|\,|z_2| \tag{10.2.24}$$

and[3]

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \tag{10.2.25}$$

In other words, the absolute value of the product is the product of the absolute values of the factors while the argument of the product is the sum of the arguments of the factors. Note also that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \tag{10.2.26}$$

[3]Since arg z is multivalued, we read Eqs. 10.2.25 and 10.2.28 as stating that there exist arguments of z_1 and z_2 for which the equality holds (see Problems 31 and 32).

Hence,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \tag{10.2.27}$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \tag{10.2.28}$$

From repeated applications of Eq. 10.2.23 we derive the rule

$$\begin{aligned} z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n[&\cos(\theta_1 + \theta_2 + \cdots + \theta_n) \\ &+ i\sin(\theta_1 + \theta_2 + \cdots + \theta_n)] \end{aligned} \tag{10.2.29}$$

An important special case occurs when $z_1 = z_2 = \cdots = z_n$. Then

$$z^n = r^n(\cos n\theta + i\sin n\theta), \quad n = 0, 1, 2, \ldots \tag{10.2.30}$$

The symbol $z^{1/n}$ expresses the statement that $(z^{1/n})^n = z$; that is, $z^{1/n}$ is an nth root of z. Equation 10.2.30 enables us to find the n nth roots of any complex number z. Let

$$z = r(\cos\theta + i\sin\theta) \tag{10.2.31}$$

For each nonnegative integer k, it is also true that

$$z = r[\cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k)] \tag{10.2.32}$$

So

$$z^{1/n} = z_k = r^{1/n}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right] \tag{10.2.33}$$

has the property that $z_k^n = z$ according to Eq. 10.2.30. It is obvious that $z_0, z_1, \ldots, z_{n-1}$ are distinct complex numbers, unless $z = 0$. Hence, for $k = 0, 1, \ldots, n - 1$, Eq. 10.2.33 provides n distinct nth roots of z.

To find n roots of $z = 1$, we define the special symbol

$$\omega_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} \tag{10.2.34}$$

Then $\omega_n, \omega_n^2, \ldots, \omega_n^n = 1$ are the n distinct roots of 1. Note that

$$\omega_n^k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n} \tag{10.2.35}$$

As a problem in the problem set, the reader is asked to prove that if $1 \le k < j \le n$, then $\omega_n^k \ne \omega_n^j$.

Now Suppose that $z_0^n = z$, so that z_0 is an nth root of z. Then the set

$$z_0, \omega_n z_0, \ldots, \omega_n^{n-1} z_0 \tag{10.2.36}$$

is the set of nth roots of z since

$$\left(\omega_n^k z_0\right)^n = \omega_n^{kn} z_0^n = 1 \cdot z \tag{10.2.37}$$

Several examples illustrate this point.

EXAMPLE 10.2.1

Express the complex number $3 + 6i$ in polar form, and also divide it by $2 - 3i$.

► Solution

To express $3 + 6i$ in polar form we must determine r and θ. We have

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 6^2} = 6.708$$

The angle θ is found, in degrees, to be

$$\theta = \tan^{-1} \tfrac{6}{3} = 63.43^\circ$$

In polar form we have

$$3 + 6i = 6.708(\cos 63.43^\circ + i \sin 63.43^\circ)$$

The desired division is

$$\frac{3+6i}{2-3i} = \frac{3+6i}{2-3i}\frac{2+3i}{2+3i} = \frac{6-18+i(12+9)}{4+9} = \frac{1}{13}(-12+21i) = -0.9231 + 1.615i$$

EXAMPLE 10.2.2

What set of points in the complex plane (i.e., the xy plane) satisfies

$$\left|\frac{z}{z-1}\right| = 2$$

► Solution

First, using Eq.10.2.27, we can write

$$\left|\frac{z}{z-1}\right| = \frac{|z|}{|z-1|}$$

Then, recognizing that the magnitude squared of a complex number is the real part squared plus the imaginary part squared, we have

$$\frac{|z|^2}{|z-1|^2} = \frac{x^2+y^2}{(x-1)^2+y^2}$$

EXAMPLE 10.2.2 *(Continued)*

where we have used $z-1 = x-1+iy$. The desired equation is then

$$\frac{x^2+y^2}{(x-1)^2+y^2} = 4$$

or

$$x^2+y^2 = 4(x-1)^2+4y^2$$

This can be written as

$$\left(x-\tfrac{4}{3}\right)^2 + y^2 = \tfrac{4}{9}$$

which is the equation of a circle of radius $\frac{2}{3}$ with center at $\left(\frac{4}{3}, 0\right)$.

EXAMPLE 10.2.3

Find the three cube roots of unity.

► Solution

These roots are ω_3, ω_3^2, 1, where

$$\omega_3 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

so

$$\omega_3^2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

This is, of course, equal to

$$\omega_3^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

We also have

$$\omega_3^3 = \cos\frac{2\pi\cdot 3}{3} + i\sin\frac{2\pi\cdot 3}{3} = \cos 2\pi + i\sin 2\pi = 1$$

EXAMPLE 10.2.4

Find the three roots of $z = -1$ using Eq. 10.2.36.

► Solution

Since $(-1)^3 = -1$, -1 is a cube root of -1. Hence, $-\omega_3$, $-\omega_3^2$, and -1 are the three distinct cube roots of -1; in the notation of Eq. 10.2.33, the roots are

$$z_0 = -1, \quad z_1 = \frac{1}{2} - i\frac{\sqrt{3}}{2}, \quad z_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

Note that

$$(-\omega_3)^3 = (-1)^3\omega_3^3 = -1$$

EXAMPLE 10.2.5

Find the three cube roots of $z = i$ using Eq. 10.2.36.

► Solution

We write i in polar form as

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

Then

$$z_0 = i^{1/3} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$$

Now the remaining two roots are $z_0\omega_3$, $z_0\omega_3^2$, or

$$z_1 = \left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$$

$$z_2 = \left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -i$$

EXAMPLE 10.2.6

Determine (a) $(3+4i)^2$ using Eq. 10.2.30, and (b) $(3+4i)^{1/3}$ using Eq. 10.2.33.

► Solution

The number is expressed in polar form, using $r=\sqrt{3^2+4^2}=5$ and $\theta=\tan^{-1}\frac{4}{3}=53.13^\circ$ as

$$3+4i=5(\cos 53.13^\circ+i\sin 53.13^\circ)$$

(a) To determine $(3+4i)^2$ we use Eq. 10.2.30 and find

$$\begin{aligned}(3+4i)^2&=5^2(\cos 2\times 53.13^\circ+i\sin 2\times 53.13^\circ)\\&=25(-0.280+0.960i)\\&=-7+24i\end{aligned}$$

We could also simply form the product

$$\begin{aligned}(3+4i)^2&=(3+4i)(3+4i)\\&=9-16+12i+12i=-7+24i\end{aligned}$$

(b) There are three distinct cube roots that must be determined when evaluating $(3+4i)^{1/3}$. They are found by expressing $(3+4i)^{1/3}$ as

$$(3+4i)^{1/3}=5^{1/3}\left(\cos\frac{53.13+360k}{3}+i\sin\frac{53.13+360k}{3}\right)$$

where the angles are expressed in degrees, rather than radians. The first root is then, using $k=0$,

$$\begin{aligned}(3+4i)^{1/3}&=5^{1/3}(\cos 17.71^\circ+i\sin 17.71^\circ)\\&=1.710(0.9526+0.3042i)\\&=1.629+0.5202i\end{aligned}$$

The second root is, using $k=1$,

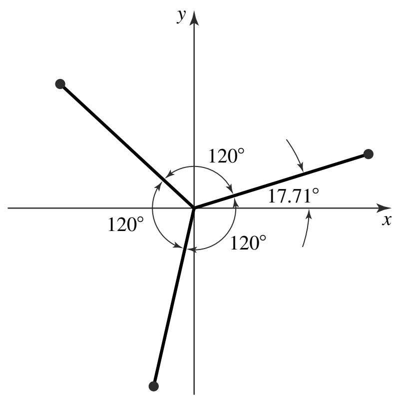
$$\begin{aligned}(3+4i)^{1/3}&=5^{1/3}(\cos 137.7^\circ+i\sin 137.7^\circ)\\&=1.710(-0.7397+0.6729i)\\&=-1.265+1.151i\end{aligned}$$

The third and final root is, using $k=2$,

$$\begin{aligned}(3+4i)^{1/3}&=5^{1/3}(\cos 257.7^\circ+i\sin 257.7^\circ)\\&=1.710(-0.2129-0.9771i)\\&=-0.3641-1.671i\end{aligned}$$

EXAMPLE 10.2.6 *(Continued)*

It is easy to verify that if we choose $k \geq 3$, we would simply return to one of the three roots already computed. The three distinct roots are illustrated in the following diagram.



10.2.1 *Maple* Applications

In *Maple,* the symbol `I` is reserved for $\sqrt{-1}$. So, in Eqs. 10.2.3:

```
>z1:=6+4*I;  z2:=6-4*I;
```

$$z1 := 6 + 4\,I$$
$$z2 := 6 - 4\,I$$

The real and imaginary parts of a complex numbers can be determined using `Re` and `Im`:

```
>Re(z1); Im(z2);
```

$$6$$
$$-4$$

The absolute value and argument of a complex number can be computed using *Maple,* via the `abs` and `argument` commands. Using z_1 of Eq.10.2.3:

```
>abs(z1); argument(z1);
```

$$2\sqrt{13}$$
$$\arctan\left(\frac{2}{3}\right)$$

The conjugate can also be determined in *Maple* for $2 + 3i$:

```
>conjugate(2+3*I);
```

$$2 - 3\,I$$

The usual symbols of $+$, $-$, etc., are used by *Maple* for complex binary operations.

Maple's `solve` command is designed to find both real and complex roots. Here we see how to find the three cube roots of unity:

```
>solve(z^3-1=0, z);
```

$$1, -\frac{1}{2}+\frac{1}{2}I\sqrt{3}, -\frac{1}{2}-\frac{1}{2}I\sqrt{3}$$

Problems

Determine the angle θ, in degrees and radians, which is necessary to write each complex number in polar form.

1. $4+3i$
2. $-4+3i$
3. $4-3i$
4. $-4-3i$

For the complex number $z = 3-4i$, find each following term.

5. z^2
6. $z\bar{z}$
7. $z/\bar{z}$
8. $\left|\frac{z+1}{z-1}\right|$
9. $(z+1)(z-i)$
10. $|z^2|$
11. $(z-i)^2/(z-1)^2$
12. z^4
13. $z^{1/2}$
14. $z^{1/3}$
15. $z^{2/3}$
16. $\frac{z^2}{z^{1/2}}$

Determine the roots of each term (express in the form $a+ib$).

17. $1^{1/5}$
18. $-16^{1/4}$
19. $i^{1/3}$
20. $9^{1/2}$

Show that each equation represents a circle.

21. $|z| = 4$
22. $|z-2| = 2$
23. $|(z-1)/(z+1)| = 3$

Find the equation of each curve represented by the following.

24. $|(z-1)/(z+1)| = 4$
25. $|(z+i)/(z-i)| = 2$

26. Identify the region represented by $|z-2| \le x$.
27. Show that for each n and each complex number $z \ne 1$,
$$1+z+z^2+\cdots+z^{n-1} = \frac{1-z^n}{1-z}$$
28. Use the result in Problem 27 to find
$$1+\omega_n+\omega_n^2+\cdots+\omega_n^{n-1}$$
where ω_n is a nonreal nth root of unity.
29. Find the four solutions of $z^4+16=0$.
30. Show geometrically why $|z_1-z_2| \ge |z_1|-|z_2|$.
31. Find arguments of $z_1 = -1+i$ and $z_2 = 1-i$ so that $\arg z_1z_2 = \arg z_1 + \arg z_2$. Explain why this equation is false if $0 \le \arg z < 2\pi$ is a requirement on z_1, z_2, and z_1z_2.
32. Find z_1 and z_2 so that $0 \le \arg z_1$, $\arg z_2 < 2\pi$, and $0 \le \arg(z_1/z_2) < 2\pi$ makes $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ false.

Use *Maple* to solve

33. Problem 5
34. Problem 6
35. Problem 7
36. Problem 8
37. Problem 9
38. Problem 10
39. Problem 11

40. Problem 12

41. Problem 13

42. Problem 14

43. Problem 15

44. Problem 16

45. Problem 17

46. Problem 18

47. Problem 19

48. Problem 20

10.3 ELEMENTARY FUNCTIONS

Most functions of a real variable which are of interest to the natural scientist can be profitably extended to a function of a complex variable by replacing the real variable x by $z = x + iy$. This guarantees that when $y = 0$, the generalized variable reduces to the original real variable. As we shall see, this simple device generates remarkable insight into our understanding of the classical functions of mathematical physics. One especially attractive example is the interconnection between the inverse tangent and the logarithm, which is presented in Eq. 10.3.25 below.

A *polynomial* is an expression

$$P_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \tag{10.3.1}$$

where the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are complex and n is a nonnegative integer. These are the simplest functions. Their behavior is well understood and easy to analyze. The next class of functions comprises the *rational* functions, the quotients of polynomials:

$$Q(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0} \tag{10.3.2}$$

The polynomial comprising the denominator of $Q(z)$ is understood to be of degree ≥ 1 so that $Q(z)$ does not formally reduce to a polynomial.

From these two classes we move to *power series,* defined as

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{10.3.3}$$

where the series is assumed convergent for all z, $|z| < R$, $R > 0$. Various tests are known that determine R. When

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{R} \tag{10.3.4}$$

exists,[4] the series in Eq. 10.3.3 converges for all z in $|z| < R$, and diverges for all z, $|z| > R$. No general statement, without further assumptions on either $f(z)$ or the sequence $a_0, a_1, a_2, \ldots,$ can be made for those z on the *circle of convergence,* $|z| = R$.

[4]This quotient in Eq. 10.3.4 is either 0 or ∞ for the series in (10.3.6) and (10.3.7). Nonetheless, these series do converge for all z.

We define e^z, $\sin z$, and $\cos z$ by the following series, each of which converges for all z:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \tag{10.3.5}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \tag{10.3.6}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \tag{10.3.7}$$

These definitions are chosen so that they reduce to the standard Taylor series for e^x, $\sin x$, and $\cos x$ when $y = 0$. The following is an elementary consequence of these formulas:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2} \tag{10.3.8}$$

Also, we note that, letting $z = i\theta$,

$$\begin{aligned} e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^4}{5!} - \cdots\right) \\ &= \cos\theta + i\sin\theta \end{aligned} \tag{10.3.9}$$

This leads to a very useful expression for the complex veriable z. In polar form, $z = r(\cos\theta + i\sin\theta)$, so Eq. 10.3.9 allows us to write

$$z = re^{i\theta} \tag{10.3.10}$$

This form is quite useful in obtaining powers and roots of z and in various other operations involving complex numbers.

The *hyperbolic sine* and *cosine* are defined as

$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^{-z}}{2} \tag{10.3.11}$$

With the use of Eqs. 10.3.8 we see that

$$\begin{aligned} \sinh iz &= i\sin z, \qquad \sin iz = i\sinh z, \\ \cosh iz &= \cos z, \qquad \cos iz = \cosh z \end{aligned} \tag{10.3.12}$$

We can then separate the real and imaginary parts from $\sin z$ and $\cos z$, with the use of trigonometric identities, as follows:

$$\begin{aligned} \sin z &= \sin(x + iy) \\ &= \sin x \cos iy + \sin iy \cos x \\ &= \sin x \cos y + i \sinh y \cos x \end{aligned} \tag{10.3.13}$$

$$\begin{aligned} \cos z &= \cos(x + iy) \\ &= \cos x \cos iy - \sin x \sin iy \\ &= \cos x \cosh y - i \sin x \sinh y \end{aligned} \tag{10.3.14}$$

The *natural logarithm* of z, written $\ln z$, should be defined so that

$$e^{\ln z} = z \tag{10.3.15}$$

Using Eq. 10.3.10, we see that

$$\begin{aligned} \ln z &= \ln(re^{i\theta}) \\ &= \ln r + i\theta \end{aligned} \tag{10.3.16}$$

is a reasonable candidate for this definition. An immediate consequence of this definition shows that

$$\begin{aligned} e^{\ln z} &= e^{\ln r + i\theta} \\ &= e^{\ln r} e^{i\theta} \\ &= r(\cos\theta + i\sin\theta) = z \end{aligned} \tag{10.3.17}$$

using Eq. 10.3.9. Hence, $\ln z$, so defined, does satisfy Eq. 10.3.15. Since θ is multivalued, we must restrict its value so that $\ln z$ becomes $\ln x$ when $y = 0$. The restrictions $0 \le \theta < 2\pi$ or $-\pi < \theta \le \pi$ are both used. The *principal value*[5] of $\ln z$ results when $0 \le \theta < 2\pi$.

We are now in a position to define z^a for any complex number a. By definition

$$z^a = e^{a \ln z} \tag{10.3.18}$$

We leave it to the reader to verify that definition 10.3.18 agrees with the definition of z^a when the exponent is a real fraction or integer (see Problem 53).

Finally, in our discussion of elementary functions, we include the inverse trigonometric functions and inverse hyperbolic functions. Let

$$w = \sin^{-1} z \tag{10.3.19}$$

Then, using Eq. 10.3.8,

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i} \tag{10.3.20}$$

Rearranging and multiplying by $2ie^{iw}$ gives

$$e^{2iw} - 2ize^{iw} - 1 = 0 \tag{10.3.21}$$

This quadratic equation (let $e^{iw} = \phi$, so that $\phi^2 - 2iz\phi - 1 = 0$) has solutions

$$e^{iw} = iz + (1 - z^2)^{1/2} \tag{10.3.22}$$

The square root is to be understood in the same sense as Section 10.2. We solve for iw in Eq. 10.3.22 and obtain

$$w = \sin^{-1} z = -i \ln[iz + (1 - z^2)^{1/2}] \tag{10.3.23}$$

This expression is double-valued because of the square root and is multi-valued because of the logarithm. Two principal values result for each complex number z except for $z = 1$, in which

[5]The principal value of $\ln z$ is discontinuous in regions containing the positive real axis because $\ln x$ is not close to $\ln(x - i\epsilon)$ for small ϵ. This is true since Im $[\ln x] = 0$ but Im $[\ln(x - i\epsilon)]$ is almost 2π. For this reason, we often define $\ln z$ by selecting $-\pi < \arg \ln z < \pi$. Then $\ln z$ is continuous for all z, excluding $-\infty < z = x \le 0$.

case the square-root quantity is zero. In a similar manner we can find expressions for the other inverse functions. They are listed in the following:

$$\begin{aligned}\sin^{-1} z &= -i \ln[iz + (1 - z^2)^{1/2}]\\ \cos^{-1} z &= -i \ln[z + (z^2 - 1)^{1/2}]\\ \tan^{-1} z &= \frac{i}{2} \ln \frac{1 - iz}{1 + iz}\\ \sinh^{-1} z &= \ln[z + (1 + z^2)^{1/2}]\\ \cosh^{-1} z &= \ln[z + (z^2 - 1)^{1/2}]\\ \tanh^{-1} z &= \frac{1}{2} \ln \frac{1 + z}{1 - z}\end{aligned} \tag{10.3.24}$$

It is worthwhile to note that the exponential and logarithmic functions are sufficient to define z^a, $\sin z$, $\cos z$, $\tan z$, $\csc z$, $\sec z$, $\sin^{-1} z$, $\cos^{-1} z$, $\tan^{-1} z$, $\sinh z$, $\cosh z$, $\tanh z$, $\sinh^{-1} z$, $\cosh^{-1} z$, and $\tanh^{-1} z$; these interconnections are impossible to discover without the notion of a complex variable. Witness, for $z = x$, that the third equation in 10.3.24 is

$$\tan^{-1} x = \frac{i}{2} \ln \frac{1 - ix}{1 + ix} \tag{10.3.25}$$

a truly remarkable formula.

EXAMPLE 10.3.1

Find the principal value of i^i.

▶ Solution

We have

$$i^i = e^{i \ln i} = e^{i[\ln 1 + (\pi/2)i]} = e^{i[(\pi/2)i]} = e^{-\pi/2}$$

a result that delights the imagination, because of the appearance of e, π, and i in one simple equation.

EXAMPLE 10.3.2

Find the principal value of

$$(2 + i)^{1-i}$$

▶ Solution

Using Eq. 10.3.18, we can write

$$(2 + i)^{1-i} = e^{(1-i) \ln(2+i)}$$

EXAMPLE 10.3.2 *(Continued)*

We find the principal value of $(2+i)^{1-i}$ by using the principal value of $\ln(2+i)$:

$$\ln(2+i) = \ln\sqrt{5} + 0.4636i$$

since $\tan^{-1}\ 1/2 = 0.4636$ rad. Then

$$\begin{aligned}(2+i)^{1-i} &= e^{(1-i)(\ln\sqrt{5}+0.4636i)} = e^{1.2683-0.3411i}\\ &= e^{1.2683}(\cos 0.3411 - i\sin 0.3411)\\ &= 3.555(0.9424 - 0.3345i)\\ &= 3.350 - 1.189i\end{aligned}$$

EXAMPLE 10.3.3

Using $z = 3-4i$, find the value or principal value of (a) e^{iz}, (b) e^{-iz}, (c) $\sin z$, and (d) $\ln z$.

▶ Solution

(a)
$$\begin{aligned}e^{i(3-4i)} &= e^{4+3i} = e^4 e^{3i}\\ &= 54.60(\cos 3 + i\sin 3)\\ &= 54.60(-0.990 + 0.1411i)\\ &= -54.05 + 7.704i\end{aligned}$$

(b)
$$\begin{aligned}e^{-i(3-4i)} &= e^{-4-3i} = e^{-4}e^{-3i}\\ &= 0.01832[\cos(-3) + i\sin(-3)]\\ &= 0.01832[-0.990 - 0.1411i]\\ &= -0.01814 - 0.002585i\end{aligned}$$

(c)
$$\begin{aligned}\sin(3-4i) &= \frac{e^{i(3-4i)} - e^{-i(3-4i)}}{2i}\\ &= \frac{-54.05 + 7.704i - (-0.01814 - 0.002585i)}{2i}\\ &= 3.853 + 27.01i\end{aligned}$$

(d)
$$\begin{aligned}\ln(3-4i) &= \ln r + i\theta\\ &= \ln 5 + i\tan^{-1}\frac{-4}{3}\\ &= 1.609 + 5.356i\end{aligned}$$

where the angle θ is expressed in radians.

EXAMPLE 10.3.4

What is the value of z so that $\sin z = 10$?

► Solution

From Eq. 10.3.24, we write

$$z = \sin^{-1} 10 = -i \ln[10i + (-99)^{1/2}]$$

The two roots of -99 are $3\sqrt{11}i$ and $-3\sqrt{11}i$. Hence,

$$z_1 = -i \ln[(10 + 3\sqrt{11})i], \qquad z_2 = -i \ln[(10 - 3\sqrt{11})i]$$

But if α is real, $\ln \alpha i = \ln|\alpha| + (\pi/2)i$. Hence,

$$z_1 = \frac{\pi}{2} - i \ln(10 + 3\sqrt{11}), \qquad z_2 = \frac{\pi}{2} - i \ln(10 - 3\sqrt{11})$$

or

$$z_1 = \frac{\pi}{2} - 2.993i, \qquad z_2 = \frac{\pi}{2} + 2.993i$$

10.3.1 *Maple* Applications

The functions described in this section are all built into *Maple,* and automatically determine complex output. To access these functions in *Maple,* use `sin`, `cos`, `sinh`, `cosh`, `exp`, and `ln`. For the inverse trigonometric and hyperbolic functions, use `arcsin`, `arccos`, `arcsinh`, and `arccosh`. *Maple* uses the principal value of $\ln z$.

At times, the `evalc` command ("evaluate complex number") is necessary to force *Maple* to find values. For instance, Example 10.3.1 would be calculated this way:

```
>I^I;
```

$$I^I$$

```
>evalc (I^I);
```

$$e^{(-\frac{\pi}{2})}$$

Example 10.3.2 can be reproduced in this way with *Maple*:

```
>(2+I)^(1-I);
```

$$(2 + I)^{(1-I)}$$

```
>evalc(%);
```

$$e^{(1/2\ln(5)+\arctan(1/2))} \cos\left(\frac{1}{2}\ln(5) - \arctan\left(\frac{1}{2}\right)\right)$$
$$-e^{(1/2\ln(5)+\arctan(1/2))} \sin\left(\frac{1}{2}\ln(5) - \arctan\left(\frac{1}{2}\right)\right) I$$

```
>evalf(%);
```

$$3.350259315 - 1.189150220\, I$$

Problems

1. Show that $\sin z = (e^{iz} - e^{-iz})/2i$ and $\cos z = (e^{iz} + e^{-iz})/2$ using Eqs. 10.3.5 through 10.3.7.

Express each complex number in exponential form (see Eq. 10.3.10).

2. -2
3. $2i$
4. $-2i$
5. $3 + 4i$
6. $5 - 12i$
7. $-3 - 4i$
8. $-5 + 12i$
9. $0.213 - 2.15i$

10. Using $z = (\pi/2) - i$, show that Eq. 10.3.8 yields the same result as Eq. 10.3.13 for $\sin z$.

Find the value of e^z for each value of z.

11. $\frac{\pi}{2}i$
12. $2i$
13. $-\frac{\pi}{4}i$
14. $4\pi i$
15. $2 + \pi i$
16. $-1 - \frac{\pi}{4}i$

Find each quantity using Eq. 10.3.10.

17. $1^{1/5}$
18. $(1 - i)^{1/4}$
19. $(-1)^{1/3}$
20. $(2 + i)^3$
21. $(3 + 4i)^4$
22. $\sqrt{2 - i}$

For the value $z = \pi/2 - (\pi/4)i$, find each term.

23. e^{iz}
24. $\sin z$
25. $\cos z$
26. $\sinh z$
27. $\cosh z$
28. $|\sin z|$
29. $|\tan z|$

Find the principal value of the $\ln z$ for each value for z.

30. i
31. $3 + 4i$
32. $4 - 3i$
33. $-5 + 12i$
34. ei
35. -4
36. e^i

Using the relationship that $z^a = e^{\ln z^a} = e^{a \ln z}$, find the principal value of each power.

37. i^i
38. $(3 + 4i)^{(1-i)}$
39. $(4 - 3i)^{(2+i)}$
40. $(1 + i)^{(1+i)}$
41. $(-1 - i)^{-i/2}$

Find the values or principal values of z for each equation.

42. $\sin z = 2$
43. $\cos z = 4$
44. $e^z = -3$
45. $\sin z = -2i$
46. $\cos z = -2$

Show that each equation is true.

47. $\cos^{-1} z = -i \ln[z + (z^2 - 1)^{1/2}]$
48. $\sinh^{-1} z = \ln[z + (1 + z^2)^{1/2}]$

For $z = 2 - i$, evaluate each function.

49. $\sin^{-1}$
50. $\tan^{-1} z$
51. $\cosh^{-1} z$

52. Using the principal value of $\ln z$, explain why $\ln 1$ and $\ln(1 - i\epsilon)$ are not close even when ϵ is very near zero.

53. In Eq. 10.3.18 set $a = n$, n a real integer. Show that $z^n = e^{n \ln z}$ is in agreement with $z^n = r^n$ $(\cos n\theta + i \sin n\theta)$.

54. In Eq. 10.3.18 set $a = p/q$, p/q a real rational number. Show that $z^{p/q} = e^{(p/q \ln z)}$ is the same set of complex numbers as

$$z^{p/q} = r^{p/q} \left[\cos \frac{p}{q}(\theta + 2\pi k) + \sin \frac{p}{q}(\theta + 2\pi k) \right]$$

Use *Maple* to solve

55. Problem 11
56. Problem 12
57. Problem 13
58. Problem 14
59. Problem 15
60. Problem 16
61. Problem 17
62. Problem 18
63. Problem 19
64. Problem 20
65. Problem 21
66. Problem 22
67. Problem 23
68. Problem 24
69. Problem 25
70. Problem 26
71. Problem 27
72. Problem 28
73. Problem 29
74. Problem 30
75. Problem 31
76. Problem 32
77. Problem 33
78. Problem 34
79. Problem 35
80. Problem 36
81. Problem 37
82. Problem 38
83. Problem 39
84. Problem 40
85. Problem 41
86. Problem 42
87. Problem 43
88. Problem 44
89. Problem 45
90. Problem 46
91. Problem 49
92. Problem 50
93. Problem 51

10.4 ANALYTIC FUNCTIONS

In Section 10.3 we motivated the definitions of the various "elementary" functions by requiring $f(z)$ to reduce to the standard function when $y = 0$. Although intuitively appealing, this consideration is only part of the picture. In this section we round out our presentation by showing that our newly defined functions satisfy the appropriate differential relationships:

$$\frac{de^z}{dz} = e^z, \qquad \frac{d \cos z}{dz} = -\sin z$$

and so on.

The definition of the derivative of a function $f(z)$ is

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{10.4.1}$$

It is important to note that in the limiting process as $\Delta z \to 0$ there are an infinite number of paths that Δz can take. Some of these are sketched in Fig. 10.3. For a derivative to exist we demand that $f'(z)$ be unique as $\Delta z \to 0$, regardless of the path chosen. In real variables this

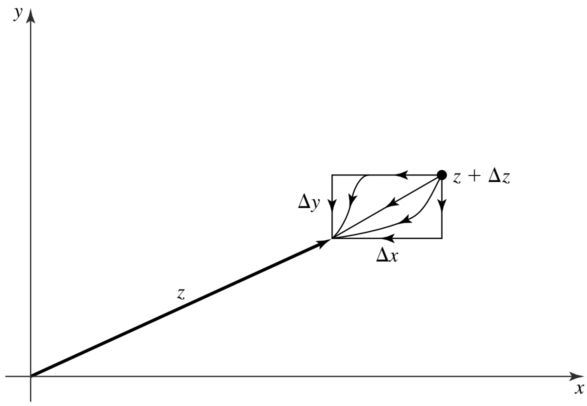


Figure 10.3 Various paths for Δz to approach zero.

restriction on the derivative was not necessary since only one path was used, along the x axis only. Let us illustrate the importance of this demand with the function

$$f(z) = \bar{z} = x - iy \tag{10.4.2}$$

The quotient in the definition of the derivative using $\Delta z = \Delta x + i\Delta y$ is

$$\begin{aligned}\frac{f(z+\Delta z) - f(z)}{\Delta z} &= \frac{[(x+\Delta x) - i(y+\Delta y)] - (x - iy)}{\Delta x + i\Delta y}\\ &= \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}\end{aligned} \tag{10.4.3}$$

First, let $\Delta y = 0$ and then let $\Delta x \to 0$. Then, the quotient is $+1$. Next, let $\Delta x = 0$ and then let $\Delta y \to 0$. Now the quotient is -1. Obviously, we obtain a different value for each path. Actually, there is a different value for the quotient for each value of the slope of the line along which Δz approaches zero (see Problem 2). Since the limit is not unique, we say that the derivative does not exist. We shall now derive conditions which must hold if a function has a derivative.

Let us assume now that the derivative $f'(z)$ does exist. The real and imaginary parts of $f(z)$ are denoted by $u(x, y)$ and $v(x, y)$, respectively; that is,

$$f(z) = u(x, y) + iv(x, y) \tag{10.4.4}$$

First, let $\Delta y = 0$ so that $\Delta z \to 0$ parallel to the x axis. From Eq. 10.4.1 with $\Delta z = \Delta x$,

$$\begin{aligned}f'(z) &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x, y) - iv(x, y)}{\Delta x}\\ &= \lim_{\Delta x \to 0}\left[\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x, y)}{\Delta x}\right]\\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\end{aligned} \tag{10.4.5}$$

Next, let $\Delta x = 0$ so that $\Delta z \to 0$ parallel to the y axis. Then, using $\Delta z = i\Delta y$,

$$\begin{aligned} f'(z) &= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - u(x, y) - iv(x, y)}{i\Delta y} \\ &= \lim_{\Delta y \to 0} \left[\frac{u(x, y+\Delta y) - u(x, y)}{i\Delta y} + \frac{v(x, y+\Delta y) - v(x, y)}{\Delta y} \right] \\ &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \end{aligned} \tag{10.4.6}$$

For the derivative to exist, it is necessary that these two expressions for $f'(z)$ be equal. Hence,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \tag{10.4.7}$$

Setting the real parts and the imaginary parts equal to each other, respectively, we find that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{10.4.8}$$

the famous *Cauchy–Riemann equations.* We have derived these equations by considering only two possible paths along which $\Delta z \to 0$. It can be shown (we shall not do so in this text) that no additional relationships are necessary to ensure the existence of the derivative. If the Cauchy–Riemann equations are satisfied at a point $z = z_0$, and the first partials of u and v are continuous at z_0, then the derivative $f'(z_0)$ exists. If $f'(z)$ exists at $z = z_0$ and at every point in a neighborhood of z_0, then the function $f(z)$ is said to be *analytic* at z_0.

Thus, the definition of analyticity puts some restrictions on the nature of the sets on which $f(z)$ is analytic.[6] For instance, if $f(z)$ is analytic for all z, $z < 1$ and at $z = i$ as well, then $f(z)$ is analytic at least in a domain portrayed in Fig. 10.4. If $f(z)$ is not analytic at z_0, $f(z)$ is *singular* at z_0. In most applications z_0 is an *isolated singular point,* by which we mean that in some neighborhood of z_0, $f(z)$ is analytic for $z \neq z_0$ and singular only at z_0. The most common singular points of an otherwise analytic function arise because of zeros in the denominator of a quotient. For each of the following functions, $f(z)$ has an isolated singular point at $z_0 = 0$:

$$\frac{1}{e^z - 1}, \quad \frac{1}{z(z+1)}, \quad \frac{1}{z \sin z}, \quad \tan z \tag{10.4.9}$$

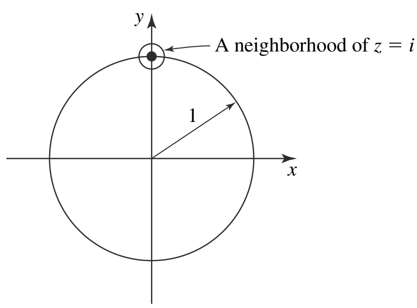


Figure 10.4 A set of analyticity for some $f(z)$.

[6]We do not explore this point here. Suffice it to say that functions are analytic on open sets in the complex plane.

The rational function

$$Q(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0} \tag{10.4.10}$$

has isolated singularities at each zero of the denominator polynomial. We use "singularity" and mean "isolated singularity" unless we explicitly comment to the contrary.

Since the definition of $f'(z)$ is formally the same as the definition of $f'(z)$, we can mirror the arguments in elementary calculus to prove that

(1)
$$\frac{d}{dz}[f(z) \pm g(z)] = f'(z) \pm g'(z) \tag{10.4.11}$$

(2)
$$\frac{d}{dz}[kf(z)] = kf'(z) \tag{10.4.12}$$

(3)
$$\frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + g'(z)f(z) \tag{10.4.13}$$

(4)
$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)} \tag{10.4.14}$$

(5)
$$\frac{d}{dz}[f(g(z))] = \frac{df}{dg}\frac{dg}{dz} \tag{10.4.15}$$

Also, we can show that

$$\begin{aligned}&\frac{d}{dz}(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0)\\ &\quad = n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \cdots + 2a_2 z + a_1\end{aligned} \tag{10.4.16}$$

by using properties (1) and (2) and the easily verified facts

$$\frac{dz}{dz} = 1, \qquad \frac{da_0}{dz} = 0 \tag{10.4.17}$$

EXAMPLE 10.4.1

Determine if and where the functions $z\bar{z}$ and z^2 are analytic.

► Solution

The function $z\bar{z}$ is written as

$$f(z) = z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$$

and is a real function only; its imaginary part is zero. That is,

$$u = x^2 + y^2, \qquad v = 0$$

EXAMPLE 10.4.1 *(Continued)*

The Cauchy–Riemann equations give

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{or} \qquad 2x = 0$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{or} \qquad 2y = 0$$

Hence, we see that x and y must each be zero for the Cauchy–Riemann equations to be satisfied. This is true at the origin but not in the neighborhood (however small) of the origin. Thus, the function $z\bar{z}$ is not analytic anywhere.

Now consider the function z^2. It is

$$f(z) = z^2 = (x + iy)(x + iy) = x^2 - y^2 + i2xy$$

The real and imaginary parts are

$$u = x^2 - y^2, \qquad v = 2xy$$

The Cauchy–Reimann equations give

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{or} \qquad 2x = 2x$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{or} \qquad -2y = -2y$$

We see that these equations are satisfied at all points in the xy plane. Hence, the function z^2 is analytic everywhere.

EXAMPLE 10.4.2

Find the regions of analyticity of the functions listed below and compute their first derivatives.
(a) e^z (b) $\ln z$ (c) $\sin z$

► Solution

(a) Since $e^z = e^{x+iy} = e^x e^{iy}$, we have

$$e^z = e^x \cos y + ie^x \sin y$$

Therefore, $u = e^x \cos y$ and $v = e^x \sin y$. The verification of the Cauchy–Riemann equations is simple, so we learn that e^z is analytic for all z. Also, from Eq. 10.4.7

$$\begin{aligned}\frac{de^z}{dz} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= e^x \cos y + ie^x \sin y = e^z\end{aligned}$$

EXAMPLE 10.4.2 *(Continued)*

(b) Here we express $\ln z = \ln r + i\theta$, $r = \sqrt{x^2 + y^2}$, and $\theta = \tan^{-1} y/x$ with $-\pi < \theta \le \pi$. Hence,

$$u = \ln \sqrt{x^2 + y^2} = \tfrac{1}{2}\ln(x^2 + y^2)$$
$$v = \tan^{-1}\frac{y}{x}$$

so

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2}$$
$$\frac{\partial v}{\partial y} = \frac{1/x}{1 + (y/x)^2} = \frac{x}{x^2 + y^2}$$

Also,

$$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{2y}{x^2 + y^2} = \frac{y}{x^2 + y^2}$$
$$\frac{\partial v}{\partial x} = \frac{-y/x^2}{1 + (y/x)^2} = -\frac{y}{x^2 + y^2}$$

The Cauchy–Riemann equations are satisfied as long as $x^2 + y^2 \neq 0$ and θ is uniquely defined, say $-\pi < \theta < \pi$. Finally,

$$\begin{aligned} \frac{d}{dz}\ln z &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} \\ &= \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} = \frac{1}{z} \end{aligned}$$

valid as long as $z \neq 0$ and $\ln z$ is continuous. For $-\pi < \theta < \pi$, $\ln z$ is continuous at every z except $z = x \le 0$.

(c) Since

$$\sin z = \sin x \cosh y + i \sin y \cos x$$

by Eq. 10.3.13, we have

$$u = \sin x \cosh y, \qquad v = \cos x \sinh y$$

so the Cauchy–Riemann equations are easily checked and are valid for all z. Moreover,

$$\begin{aligned} \frac{d}{dz}\sin z &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} \\ &= \cos x \cosh y - i \sin x \sinh y = \cos z \end{aligned}$$

from Eq. 10.3.14.

EXAMPLE 10.4.3

Find $d/dz \tan^{-1} z$.

► Solution

Since we know, by Eq. 10.3.24, that

$$\tan^{-1} z = \frac{i}{2} \ln \frac{1-iz}{1+iz}$$

we have

$$\frac{d}{dz} \tan^{-1} z = \frac{i}{2} \frac{1+iz}{1-iz} \frac{d}{dz} \left(\frac{1-iz}{1+iz} \right)$$

by utilizing $d/dz \ln z = 1/z$ and the chain rule, property (5). Since

$$\begin{aligned} \frac{d}{dz} \frac{1-iz}{1+iz} &= \frac{(-i)(1+iz) - i(1-iz)}{(1+iz)^2} \\ &= -\frac{2i}{(1+iz)^2} \end{aligned}$$

we have

$$\begin{aligned} \frac{d}{dz} \tan^{-1} z &= \frac{i}{2} \frac{1+iz}{1-iz} \frac{-zi}{(1+iz)^2} \\ &= \frac{1}{(1-iz)(1+iz)} = \frac{1}{1+z^2} \end{aligned}$$

This result could be obtained somewhat more easily by assuming that

$$\ln \frac{1-iz}{1+iz} = \ln(1-iz) - \ln(1+iz)$$

But this is not a true statement without some qualifications.[7]

[7]See Problem 16 to see one difficulty with the "rule" $\ln z_1 z_2 = \ln z_1 + \ln z_2$.

10.4.1 Harmonic Functions

Consider, once again, the Cauchy–Riemann equations 10.4.8, from which we can deduce

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} \tag{10.4.18}$$

and[8]

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y}, \qquad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \tag{10.4.19}$$

From Eqs. 10.4.18 we see that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = 0 \tag{10.4.20}$$

and from Eqs. 10.4.19,

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2} = 0 \tag{10.4.21}$$

The real and imaginary parts of an analytic function satisfy Laplace's equation. Functions that satisfy Laplace's equation are called *harmonic functions.* Hence, $u(x, y)$ and $v(x, y)$ are harmonic functions. Two functions that satisfy Laplace's equation and the Cauchy–Riemann equations are known as *conjugate harmonic functions.* If one of the conjugate harmonic functions is known, the other can be found by using the Cauchy–Riemann equations. This will be illustrated by an example.

Finally, let us show that constant u lines are normal to constant v lines if $u + iv$ is an analytic function. From the chain rule of calculus

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \tag{10.4.22}$$

Along a constant u line, $du = 0$. Hence,

$$\left.\frac{dy}{dx}\right|_{u=C} = -\frac{\partial u/\partial x}{\partial u/\partial y} \tag{10.4.23}$$

Along a constant v line, $dv = 0$, giving

$$\left.\frac{dy}{dx}\right|_{v=C} = -\frac{\partial v/\partial x}{\partial v/\partial y} \tag{10.4.24}$$

But, using the Cauchy–Riemann equations,

$$-\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial v/\partial y}{\partial v/\partial x} \tag{10.4.25}$$

The slope of the constant u line is the negative reciprocal of the slope of the constant v line. Hence, the lines are orthogonal. This property is useful in sketching constant u and v lines, as in fluid fields or electrical fields.

---

[8]We have interchanged the order of differentiation since the second partial derivatives are assumed to be continuous.

EXAMPLE 10.4.4

The real function $u(x, y) = Ax + By$ obviously satisfies Laplace's equation. Find its conjugate harmonic function, and write the function $f(z)$.

► Solution

The conjugate harmonic function, denoted $v(x, y)$, is related to $u(x, y)$ by the Cauchy–Riemann equations. Thus,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{or} \quad \frac{\partial v}{\partial y} = A$$

The solution for v is

$$v = Ay + g(x)$$

where $g(x)$ is an unknown function to be determined. The relationship above must satisfy the other Cauchy–Riemann equation.

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{or} \quad \frac{\partial g}{\partial x} = -B$$

Hence,

$$g(x) = -Bx + C$$

where C is a constant of integration, to be determined by an imposed condition. Finally, the conjugate harmonic function $v(x, y)$ is

$$v(x, y) = Ay - Bx + C$$

for every choice of C. The function $f(z)$ is then

$$\begin{aligned} f(z) &= Ax + By + i(Ay - Bx + C) \\ &= A(x + iy) - Bi(x + iy) + iC \\ &= (A - iB)z + iC \\ &= K_1 z + K_2 \end{aligned}$$

where K_1 and K_2 are complex constants.

10.4.2 A Technical Note

The definition of analyticity requires that the derivative of $f(z)$ exist in some neighborhood of z_0. It does not, apparently, place any restrictions on the behavior of this derivative. It is possible to prove[9] that $f'(z)$ is far from arbitrary; it is also analytic at z_0. This same proof applies to $f'(z)$ and leads to the conclusion that $f''(z)$ is also analytic at z_0. We have, therefore,

[9] See any text on the theory of complex variables.

Theorem 10.1: *If f is analytic at z_0, then so are $f'(z)$ and all the higher order derivatives.*

This theorem is supported by all of the examples we have studied. The reader should note that the analogous result for functions of a real variable is false (see Problem 17).

10.4.3 *Maple* Applications

Example 10.4.1 can be done with *Maple,* but we must keep in mind that in these problems, we convert the problem using real variables *x* and *y*, and *Maple* does not know that typically $z = x + iy$, so we have to make that a definition. To begin:

```
>z:= x + I*y;
```

$$z := x + yI$$

```
>z*conjugate(z);
```

$$(x + yI)\,\overline{(x + yI)}$$

```
>f:=evalc(%);
```

$$f := x^2 + y^2$$

```
>u:=evalc(Re(f)); v:=evalc(Im(f));
```

$$u := x^2 + y^2$$
$$v := 0$$

Now, if we wish, we can use the `diff` command to write the Cauchy–Riemann equations:

```
>CR1:=diff(u, x)=diff(v,y); CR2:=diff(u, y)=diff(v,x);
```

$$CR1 := 2x = 0$$
$$CR2 := 2y = 0$$

Similarly, the second part of Example 10.4.1 can be calculated in this way:

```
>f:=evalc(z*z);
```

$$f := x^2 + 2\,Ixy - y^2$$

```
>u:=evalc(Re(f)); v:=evalc(Im(f));
```

$$u := x^2 - y^2$$
$$v := 2yx$$

```
>CR1:=diff(u, x)=diff(v,y); CR2:=diff(u, y)=diff(v,x);
```

$$CR1 := 2x = 2x$$
$$CR2 := -2y = 2y$$

Note: If we still have `z` assigned as `x + I*y` in *Maple,* then entering the function of Example 10.4.3 into *Maple* would yield:

```
>diff(arctan(z), z);
  Error, Wrong number (or type) of parameters in function diff
```

However, if we unassign `z`, *Maple* produces the following:

```
>unassign('z');
>diff(arctan(z), z);
```

$$\frac{1}{1+z^2}$$

Now, *Maple* doesn't assume that z is a complex variable, so the calculation above is valid whether z is real or complex. In fact, recall that for a real variable x, the derivative of $\tan^{-1} x$ is $1/(1+x^2)$.

Problems

Compare the derivative of each function $f(z)$ using Eq. 10.4.5 with that obtained using Eq. 10.4.6.

1. z^2

2. $\bar{z}$

3. $\dfrac{1}{z+2}$

4. $(z-1)^2$

5. $\ln(z-1)$

6. e^z

7. $\bar{z}z$

8. Express a complex fraction in polar coordinates as $f(z) = u(r,\theta) + iv(r,\theta)$ and show that the Cauchy–Riemann equations can be expressed as

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Hint: Sketch Δz using polar coordinates; then, note that for $\Delta\theta = 0$, $\Delta z = \Delta r(\cos\theta + i\sin\theta)$, and for $\Delta z = 0$, $\Delta z = r\Delta\theta(-\sin\theta + i\cos\theta)$.

9. Derive Laplace's equation for polar coordinates.

10. Find the conjugate harmonic function associated with $u(r,\theta) = \ln r$. Sketch some constant u and v lines.

Show that each function is harmonic and find the conjugate harmonic function. Also, write the analytic function $f(z)$.

11. xy

12. $x^2 - y^2$

13. $e^y \sin x$

14. $\ln(x^2 + y^2)$

15. If v is a harmonic conjugate of u, find a harmonic conjugate of v.

16. Suppose that $\ln z = \ln r + i\theta$, $0 \le \theta < 2\pi$. Show that $\ln(-i)(-i) \neq \ln(-i) + \ln(-i)$. Hence, $\ln z_1 z_2 = \ln z_1 + \ln z_2$ may be false for the principal value of $\ln z$. Note that $\ln z_1 z_2 = \ln z_1 + \ln z_2 + 2\pi ni$, for some integer n.

17. Let $f(x) = x^2$ if $x \ge 0$ and $f(x) = -x^2$ if $x < 0$. Show that $f'(x)$ exists and is continuous, but $f''(x)$ does not exist at $x = 0$.

Use *Maple* to compute each derivative, and compare with your results from Problems 1–7.

18. Problem 1

19. Problem 2

20. Problem 3

21. Problem 4

22. Problem 5

23. Problem 6

24. Problem 7

25. ***Computer Laboratory Activity:*** Analytic functions map the complex plane into the complex plane, so they are difficult to visualize. However, there are graphs that can be made to help in understanding analytic functions. The idea is to create a region in the complex plane, and then determine the image of that region under the analytic function.

(a) Consider the function $f(z) = z^2$. Write $z = x + iy$, and then determine z^2 in terms of x and y. Write your answer in the form $a + bi$.

(b) Use the result of part (a) to write a new function $F(x, y) = (a, b)$, where a and b depend on x and y.
(c) Consider the square in the plane with these four corners: (1, 2), (1, 5), (4, 5), (4, 2). Determine what F of part (b) does to each of these points.
(d) Determine what F of part (b) does to each of the four sides of the square. Then create a plot of the result. Is the result a square?
(e) Follow the same steps for another square of your choice and for a triangle of your choice.

10.5 COMPLEX INTEGRATION

10.5.1 Arcs and Contours

A *smooth arc* is the set of points (x, y) that satisfy

$$x = \phi(t), \qquad y = \psi(t), \qquad a \le t \le b \tag{10.5.1}$$

where $\phi'(t)$ and $\psi'(t)$ are continuous in $[a, b]$ and do not vanish simultaneously. The circle, $x^2 + y^2 = 1$, is represented parametrically by

$$x = \cos t, \qquad y = \sin t, \qquad 0 \le t \le 2\pi \tag{10.5.2}$$

This is the most natural illustration of this method of representing a smooth arc in the xy plane. The representation

$$x = t, \qquad y = t^2, \qquad -\infty < t < \infty \tag{10.5.3}$$

defines the parabola $y = x^2$. Note that a parametric representation provides an ordering to the points on the arc. A smooth arc has length given by

$$L = \int_a^b \sqrt{[\phi'(t)]^2 + [\psi'(t)]^2}\, dt \tag{10.5.4}$$

A *contour* is a continuous chain of smooth arcs. Figure 10.5 illustrates a variety of contours.

A *simply closed contour,* or a *Jordan curve,* is a contour which does not intersect itself except that $\phi(a) = \phi(b)$ and $\psi(a) = \psi(b)$. A simply closed contour divides a plane into two parts, an "inside" and an "outside," and is traversed in the *positive sense* if the inside is to the left. The square portrayed in Fig. 10.5a is being traversed in the positive sense, as indicated by the direction arrows on that simply closed contour.

Circles in the complex plane have particularly simple parametric representations, which exploit the polar and exponential forms of z. The circle $|z| = a$ is given parametrically by

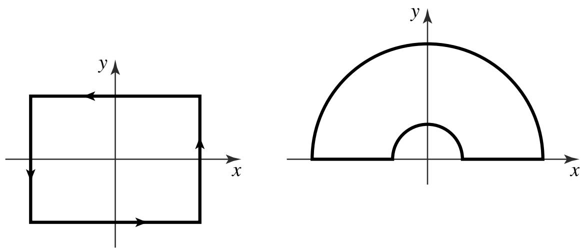
$$z = a\cos\theta + ia\sin\theta, \qquad 0 \le \theta \le 2\pi \tag{10.5.5}$$

or

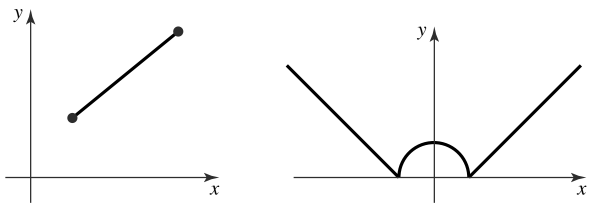
$$z = ae^{i\theta}, \qquad 0 \le \theta \le 2\pi \tag{10.5.6}$$

Both formulas are to be understood in this sense: $x = a\cos\theta$, $y = a\sin\theta$, so that $z = x + iy = a\cos\theta + ia\sin\theta = ae^{i\theta}$.

Figure 10.5 Examples of contours.



(**a**) Simply closed contours



(**b**) Contours that are not closed

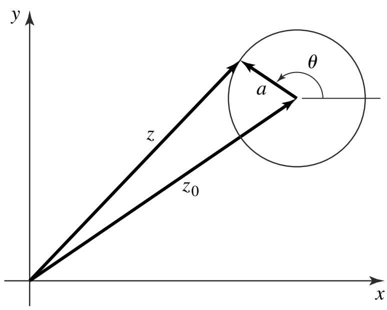


Figure 10.6 The description of a circle.

A circle of radius a, centered at z_0, as shown in Fig. 10.6, is described by the equations

$$z - z_0 = ae^{i\theta} = a \cos \theta + ia \sin \theta \tag{10.5.7}$$

where θ is measured from an axis passing through the point $z = z_0$ parallel to the x axis.

10.5.2 Line Integrals

Let z_0 and z_1 be two points in the complex plane and C a contour connecting them, as shown in Fig. 10.7. We suppose that C is defined parametrically by

$$x = \phi(t), \qquad y = \psi(t) \tag{10.5.8}$$

so that

$$z_0 = \phi(a) + i\psi(a), \qquad z_1 = \phi(b) + i\psi(b) \tag{10.5.9}$$

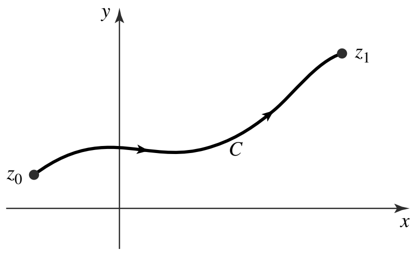


Figure 10.7 The contour C joining z_0 to z_1.

The *line integral*

$$\int_C f(z)\,dz = \int_{z_0}^{z_1} f(z)\,dz \tag{10.5.10}$$

is defined by the real integrals

$$\begin{aligned}\int_C f(z)\,dz &= \int_C (u + iv)(dx + i\,dy)\\ &= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)\end{aligned} \tag{10.5.11}$$

where we have written $f(z) = u + iv$. The integral relation 10.5.11 leads to several "natural conclusions": First,

$$\int_{z_0}^{z_1} f(z)\,dz = -\int_{z_1}^{z_0} f(z)\,dz \tag{10.5.12}$$

where the path of integration for the integral on the right-hand side of Eq. 10.5.12 is the same as that on the left but traversed in the opposite direction. Also,

$$\int_{z_0}^{z} kf(z)\,dz = k\int_{z_0}^{z} f(z)\,dz \tag{10.5.13}$$

If the contour C is a continuous chain of contours $C_1, C_2, \ldots, C_k$, such as displayed in Fig. 10.8, then

$$\int_{z_0}^{z_1} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_k} f(z)\,dz \tag{10.5.14}$$

Equation 10.5.11 can also be used to prove a most essential inequality:

Theorem 10.2: *Suppose that $|f(z)| \le M$ along the contour C and the length of C is L; then*

$$\left|\int_C f(z)\,dz\right| \le ML \tag{10.5.15}$$

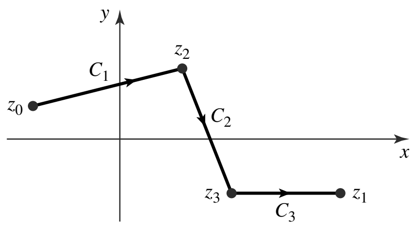


Figure 10.8 A chain of contours.

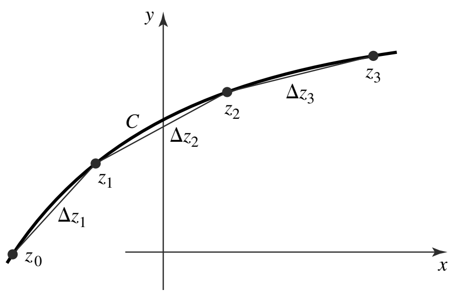


Figure 10.9 A polygonal approximation to C.

A proof of this inequality can be found in a text on complex variables. Here we outline a heuristic argument based on approximating the line integral by a sum. Consider the contour shown in Fig. 10.9 and the chords joining points $z_0, z_1, \ldots, z_n$ on C. Suppose that N is very large and the points $z_1, z_2, \ldots, z_n$ are quite close. Then

$$\int_C f(z)\,dz \cong \sum_{n=1}^{N} f(z_n)\,\Delta z_n \tag{10.5.16}$$

where $\Delta z_n = z_n - z_{n-1}$. From repeated use of $|z_1 + z_2| \leq |z_1| + |z_2|$ and Eq. 10.2.20, we have

$$\left|\int_C f(z)\,dz\right| \cong \left|\sum_{n=1}^{N} f(z_n)\,\Delta z_n\right| \leq \sum_{n=1}^{N} |f(z_n)|\,|\Delta z_n| \tag{10.5.17}$$

Now $|f(z)| \leq M$ and C, so

$$\left|\int_C f(z)\,dz\right| \leq M \sum_{n=1}^{N} |\Delta z_n| \leq ML \tag{10.5.18}$$

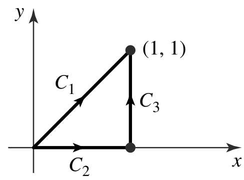
since

$$\sum_{n=1}^{N} |\Delta z_n| \cong L \tag{10.5.19}$$

When the path of integration is a simply closed contour traversed positively, we write the line integral as $\oint f(z)\,dz$; this signals an integration once around the contour in the positive sense.

EXAMPLE 10.5.1

Find the value of $\int_0^{1+i} z^2\,dz$ along the following contours. (a) The straight line from 0 to $1+i$. (b) The polygonal line from 0 to 1 and from 1 to $1+i$. The contours are sketched in the figure.



► Solution

Along any contour the integral can be written as

$$\begin{aligned}\int_0^{i+i} z^2\,dz &= \int_0^{1+i} [(x^2-y^2)+2xyi](dx+i\,dy)\\ &= \int_0^{1+i} [(x^2-y^2)\,dx - 2xy\,dy] + i\int_0^{1+i}[2xy\,dx + (x^2-y^2)\,dy]\end{aligned}$$

(a) The contour C_1 is the straight line from 0 to $1+i$ and it has the parametric representation:

$$x = t, \qquad y = t, \qquad 0 \le t \le 1$$

with $dt = dx = dy$. We make these substitutions in the integral above to obtain

$$\begin{aligned}\int_0^{1+i} z^2\,dz &= \int_0^1 [\cancelto{0}{(t^2 - t^2)}\,dt - 2t^2\,dt] + i\int_0^1 [2t^2\,dt + \cancelto{0}{(t^2 - t^2)}\,dt]\\ &= -\tfrac{2}{3} + \tfrac{2}{3}i\end{aligned}$$

(b) In this case, the contour is a polygonal line and this line requires two separate parameterizations. Using $z = x$ along C_2 and $z = 1 + iy$ along C_3, we can write

$$\int_0^{1+i} z^2\,dz = \int_0^1 x^2\,dx + \int_0^1 (1+iy)^2 i\,dy$$

This simplification follows because the contour C_2 has $y = 0$ and $dy = 0$. The contour C_3 requires $dx = 0$. Therefore,

$$\begin{aligned}\int_0^{1+i} z^2\,dz &= \frac{1}{3} + \int_0^1 (1 - y^2 + 2yi)i\,dy\\ &= \frac{1}{3} - \int_0^1 2y\,dy + i\int_0^1 (1-y^2)\,dy\\ &= \frac{1}{3} - 1 + \frac{2}{3}i = -\frac{2}{3} + \frac{2}{3}i\end{aligned}$$

EXAMPLE 10.5.2

Evaluate $\oint dz/z$ around the unit circle with center at the origin.

► Solution

The simplest representation for this circle is the exponential form

$$z = e^{i\theta} \qquad \text{and} \qquad dz = ie^{i\theta}d\theta$$

where we have noted that $r = 1$ for a unit circle with center at the origin. We then have

$$\oint \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{i\theta}d\theta}{e^{i\theta}} = \int_0^{2\pi} i d\theta = 2\pi i$$

This is an important integration technique and an important result which will be used quite often in the remainder of this chapter.

EXAMPLE 10.5.3

Evaluate the integral $\oint dz/z^n$ around the unit circle with center at the origin. Assume that n is a positive integer greater than unity.

► Solution

As in Example 10.5.2, we use $r = 1$ and the exponential form for the parametric representation of the circle:

$$z = e^{i\theta}, \qquad dz = ie^{i\theta}d\theta$$

We then have, if $n > 1$,

$$\begin{aligned} \oint \frac{dz}{z^n} &= \int_0^{2\pi} \frac{ie^{i\theta}}{e^{ni\theta}}\, d\theta \\ &= i\int_0^{2\pi} e^{i\theta(1-n)}d\theta \\ &= \left. \frac{ie^{i\theta(1-n)}}{i(1-n)} \right|_0^{2\pi} = \frac{1}{1-n}(1-1) = 0 \end{aligned}$$

EXAMPLE 10.5.4

Show that

$$\int_C dz = \int_{z_0}^{z_1} dz = z_1 - z_0$$

EXAMPLE 10.5.4 *(Continued)*

► Solution

Since $f(z) = 1$, then $u = 1$, $v = 0$ and we have, using Eq. 10.5.11,

$$\int_{z_0}^{z_1} dz = \int_{z_0}^{z_1} dx + i \int_{z_0}^{z_1} dy$$

Now suppose that C has the parametric representation

$$x = \phi(t), \qquad y = \psi(t), \qquad a \le t \le b$$

Then $dx = \phi'\, dt$, $dy = \psi'\, dt$ and

$$\begin{aligned}\int_{z_0}^{z_1} dz &= \int_a^b \phi'(t)\, dt + i \int_a^b \psi'(t)\, dt \\ &= [\phi(b) - \phi(a)] + i[\psi(b) - \psi(a)] \\ &= [\phi(b) + i\psi(b)] - [\phi(a) + i\psi(a)] \\ &= z_1 - z_0\end{aligned}$$

Note that this result is independent of the contour.

10.5.3 Green's Theorem

There is an important relationship that allows us to transform a line intergral into a double integral for contours in the xy plane. It is often referred to as *Green's theorem:*

Theorem 10.3: *Suppose that C is a simply closed contour traversed in the positive direction and bounding the region R. Suppose also that u and v are continuous with continuous first partial derivatives in R. Then*

$$\oint_C u\, dx - v\, dy = -\iint_R \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dx\, dy \tag{10.5.20}$$

Proof: Consider the curve C surrounding the region R in Fig. 10.10. Let us investigate the first part of the double integral in Green's theorem. It can be written as

$$\begin{aligned}\iint_R \frac{\partial v}{\partial x}\, dx\, dy &= \int_{h_1}^{h_2} \int_{x_1(y)}^{x_2(y)} \frac{\partial v}{\partial x}\, dx\, dy \\ &= \int_{h_1}^{h_2} [v(x_2, y) - v(x_1, y)\, dy \\ &= \int_{h_1}^{h_2} v(x_2, y)\, dy + \int_{h_2}^{h_1} v(x_1, y)\, dy\end{aligned} \tag{10.5.21}$$

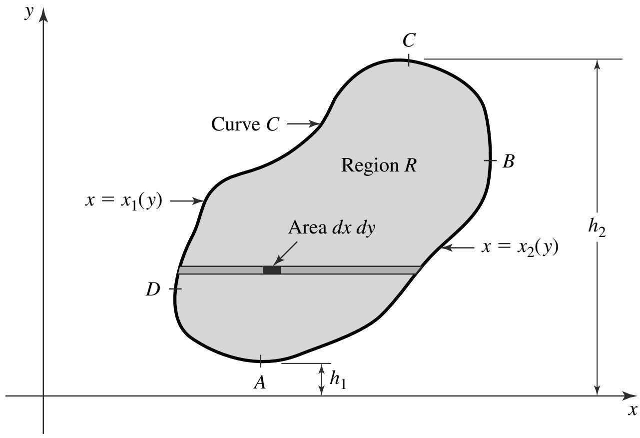


Figure 10.10 Curve C surrounding region R in Green's theorem.

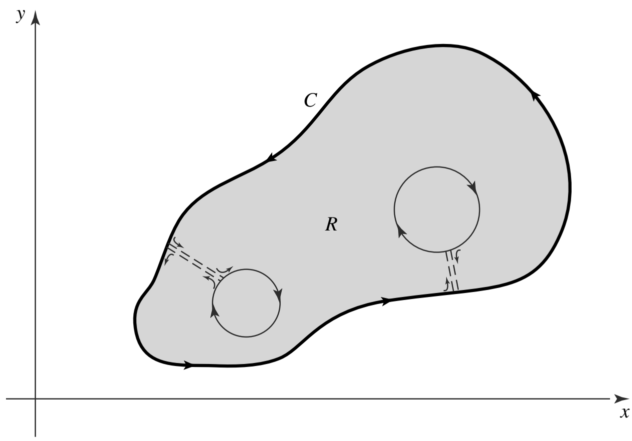


Figure 10.11 Multiply connected region.

The first integral on the right-hand side is the line integral of $v(x, y)$ taken along the path ABC from A to C and the second integral is the line integral of $v(x, y)$ taken along the path ADC from C to A. Note that the region R is on the left. Hence, we can write

$$\iint_R \frac{\partial v}{\partial x}\,dx\,dy = \oint_C v(x, y)\,dy \tag{10.5.22}$$

Similarly, we can show that

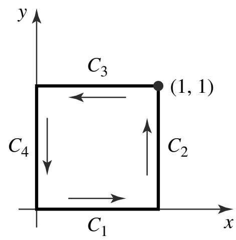
$$\iint_R \frac{\partial u}{\partial y}\,dx\,dy = -\oint_C u(x, y)\,dx \tag{10.5.23}$$

and Green's theorem is proved.

It should be noted that Green's theorem may be applied to a multiply connected region by appropriately cutting the region, as shown in Fig. 10.11. This makes a simply connected region[10] from the original multiply connected region. The contribution to the line integrals from the cuts is zero, since each cut is traversed twice, in opposite directions.

EXAMPLE 10.5.5

Verify Green's theorem by integrating the quantities $u = x + y$ and $v = 2y$ around the unit square shown.



► Solution

Let us integrate around the closed curve C formed by the four sides of the squares. We have

$$\begin{aligned}\oint u\,dx - v\,dy &= \int_{C_1} u\,dx - v\,dy + \int_{C_2} u\,dx - v\,dy \\ &\quad + \int_{C_3} u\,dx - v\,dy + \int_{C_4} u\,dx - v\,dy \\ &= \int_0^1 x\,dx + \int_0^1 -2y\,dy + \int_1^0 (x+1)\,dx + \int_1^0 -2y\,dy\end{aligned}$$

where along C_1, $dy = 0$ and $y = 0$; along C_2, $dx = 0$; along C_3, $dy = 0$ and $y = 1$; and along C_4, $dx = 0$. The equation above is integrated to give

$$\oint u\,dx - v\,dy = \tfrac{1}{2} - 1 - \left(\tfrac{1}{2} + 1\right) + 1 = -1$$

Now, using Green's theorem, let us evaluate the double integral

$$-\iint \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy$$

Using $\partial v/\partial x = 0$ and $\partial u/\partial y = 1$, there results

$$-\iint \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy = -\iint (1)\,dx\,dy = -\text{area} = -1$$

For the functions $u(x, y)$ and $v(x, y)$ of this example we have verified Green's theorem.

[10]A simply connected region is one in which any closed curve contained in the region can be shrunk to zero without passing through points not in the region. A circular ring (like a washer) is not simply connected. A region that is not simply connected is multiply connected.

10.5.4 *Maple* Applications

Contour integration is not built into *Maple,* in part because it would be difficult to define a contour in a command line. However, *Maple* can easily compute the definite integrals that arise in Example 10.5.1:

```
>int(2*y, y=0..1);  int(1-y^2, y=0..1);
```

$$1$$

$$\frac{2}{3}$$

Problems

Find convenient parametric representation of each equation.

1. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$
2. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
3. $y = 2x - 1$

Integrate each function around the closed curve indicated and compare with the double integral of Eq. 10.5.20.

4. $u = y, v = x$ around the unit square as in Example 10.5.5.
5. $u = y, v = -x$ around the unit circle with center at the origin.
6. $u = x^2 - y^2, v = -2xy$ around the triangle with vertices at (0, 0) (2, 0), (2, 2).
7. $u = x + 2y, v = x^2$ around the triangle of Problem 6.
8. $u = y^2, v = -x^2$ around the unit circle with center at the origin.

To show that line integrals are, in general, dependent on the limits of integration, evaluate each line integral.

9. $\displaystyle\int_{0,0}^{2,2} (x - iy)\,dz$ along a straight line connecting the two points.

   $\displaystyle\int_{0,0}^{2,2} (x - iy)\,dz$ along the x axis to the point (2, 0) and then vertically to (2, 2).

10. $\displaystyle\int_{0,0}^{0,2} (x^2 + y^2)\,dz$ along the y axis.

    $\displaystyle\int_{0,0}^{0,2} (x^2 + y^2)\,dz$ along the x axis to the point (2, 0), then along a circular arc.

To verify that the line integral of an analytic function is independent of the path, evaluate each line integral.

11. $\displaystyle\int_{0,0}^{2,2} z\,dz$ along a straight line connecting the two points.

    $\displaystyle\int_{0,0}^{2,2} z\,dz$ along the z axis to the point (2, 0) and then vertical.

12. $\displaystyle\int_{0,0}^{0,2} z^2\,dz$ along the x axis to the point (2, 0) and then along a circular arc.

    $\displaystyle\int_{0,0}^{0,2} z^2\,dz$ along the y axis.

Which of the following sets are simply connected?

13. The xy plane.
14. All z except z negative.
15. $|z| > 1$.
16. $0 < |z| < 1$.
17. $\operatorname{Re} z \geq 0$.
18. $\operatorname{Re} z \geq 0$ and $\operatorname{Im} z \geq 0$.
19. All z such that $0 < \arg z < \pi/4$.
20. All z such that $0 < \arg z < \pi/4$ and $|z| > 1$.
21. $\operatorname{Im} z > 0$ and $|z| > 1$. (Compare with Problem 14.)

10.6 CAUCHY'S INTEGRAL THEOREM

Now let us investigate the line integral $\oint_C f(z)\,dz$, where $f(z)$ is an analytic function within a simply connected region R enclosed by the simply closed contour C. From Eq. 10.5.11, we have

$$\oint_C f(z)\,dz = \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy) \tag{10.6.1}$$

which we have used as the definition of $\oint f(z)\,dz$. Green's theorem allows us to transform Eq. 10.6.1 into

$$\oint_C f(z)\,dz = -\iint_R \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy - i\iint_R \left(-\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dx\,dy \tag{10.6.2}$$

Using the Cauchy–Reimann equations 10.4.8, we arrive at *Cauchy's integral theorem,*

$$\oint_C f(z)\,dz = 0$$

We present it as a theorem:

Theorem 10.4: *Let C be a simply closed contour enclosing a region R in which $f(z)$ is analytic. Then*

$$\oint_C f(z)\,dz = 0 \tag{10.6.3}$$

If we divide the closed curve C into two parts, as shown in Fig. 10.12, Cauchy's integral theorem can be written as

$$\begin{aligned}\oint_C f(z)\,dz &= \underset{\text{along } C_1}{\int_a^b} f(z)\,dz + \underset{\text{along } C_2}{\int_b^a} f(z)\,dz \\ &= \underset{\text{along } C_1}{\int_a^b} f(z)\,dz - \underset{\text{along } C_2}{\int_b^a} f(z)\,dz \end{aligned} \tag{10.6.4}$$

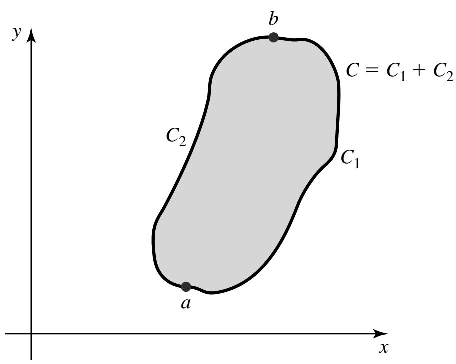


Figure 10.12 Two paths from a to b enclosing a simply connected region.

where we have reversed the order of the integration (i.e., the direction along the contour). Thus, we have

$$\underset{\text{along } C_1}{\int_a^b} f(z)\, dz = \underset{\text{along } C_2}{\int_a^b} f(z)\, dz \tag{10.6.5}$$

showing that the value of a line integral between two points is independent of the path provided that the $f(z)$ is analytic throughout a region containing the paths. In Fig. 10.12, it is sufficient to assume that $f(z)$ is analytic in the first quadrant, for example. In Example 10.5.1 we found that

$$\int_0^{1+i} z^2\, dz = -\tfrac{2}{3} + \tfrac{2}{3}i$$

regardless of whether the integration is taken along the line joining 0 to $1+i$ or the polygonal line joining 0 to 1 and then to $1+i$. Since z^2 is analytic everywhere, this result is a consequence of Eq. 10.6.5. Indeed, we can assert that this integral is independent of the path from (0, 0) to (1, 1).

10.6.1 Indefinite Integrals

The indefinite integral

$$F(z) = \int_{z_0}^{z} f(w)\, dw \tag{10.6.6}$$

defines F as a function of z as long as the contour joining z_0 to z lies entirely within a domain D which is simply connected and within which $f(z)$ is analytic. As we might reasonably expect

$$F'(z) = f(z), \qquad \text{in } D \tag{10.6.7}$$

which means that $F(z)$ itself is analytic in D. To see how this comes about, consider the difference quotient

$$\begin{aligned}\frac{\Delta F}{\Delta z} &= \frac{F(z+\Delta z) - F(z)}{\Delta z} \\ &= \frac{1}{\Delta z}\left(\int_{z_0}^{z+\Delta z} f(w)\, dw - \int_{z_0}^{z} f(w)\, dw\right) \\ &= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)\, dw\end{aligned} \tag{10.6.8}$$

Also, we can write

$$f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)\, dw \tag{10.6.9}$$

which follows from Eq. 10.5.13 and Example 10.5.4. We subtract Eq. 10.6.9 from Eq. 10.6.8 to obtain

$$\frac{\Delta F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(w) - f(z)]\, dw \tag{10.6.10}$$

Since the integral in Eq. 10.6.10 is independent of the path between z and $z + \Delta z$ we take this path as linear. As $\Delta z \to 0$, $f(w) \to f(z)$ and hence, for any $\epsilon > 0$, we can be assured that

$$|f(w) - f(z)| \le \epsilon \qquad \text{for } |\Delta z| \text{ small} \tag{10.6.11}$$

Thus,

$$\begin{aligned} \left| \frac{\Delta F(z)}{\Delta z} - f(z) \right| &= \frac{1}{|\Delta z|} \left| \int_z^{z+\Delta z} [f(w) - f(z)]\, dw \right| \\ &\le \frac{|\Delta z|}{|\Delta z|} \epsilon = \epsilon \end{aligned} \tag{10.6.12}$$

from Theorem 10.2. Clearly, $\epsilon \to 0$ as $\Delta z \to 0$. Hence,

$$\lim_{\Delta z \to 0} \frac{\Delta F(z)}{\Delta z} = F'(z) \tag{10.6.13}$$

by definition and $F'(z) = f(z)$ from Eq. 10.6.12.

The identity

$$\begin{aligned} \int_a^b f(z)\, dz &= \int_{z_0}^b f(z)\, dz - \int_{z_0}^a f(z)\, dz \\ &= F(b) - F(a) \end{aligned} \tag{10.6.14}$$

is the familiar formula from elementary calculus. The importance of Eq. 10.6.14 is this: The contour integral $\int_a^b f(z)\, dz$ may be evaluated by finding an antiderivative $F(z)$ (a function satisfying $F' = f$) and computing $[F(b) - F(a)]$ instead of parameterizing the arc joining a to b and evaluating the resulting real integrals. Compare the next example with Example 10.5.1.

EXAMPLE 10.6.1

Evaluate the integral

$$\int_0^{1+i} z^2\, dz$$

► Solution

Let

$$F(z) = \frac{z^3}{3}$$

EXAMPLE 10.6.1 *(Continued)*

Then, since $F'(z) = z^2$, we have

$$\int_0^{1+i} z^2\,dz = F(1+i) - F(0)$$
$$= \frac{(1+i)^3}{3} - 0 = -\frac{2}{3} + \frac{2}{3}i$$

as expected from Example 10.5.1. Note the more general result:

$$\int_0^z w^2\,dw = F(z) - F(0) = \frac{z^3}{3}$$

for each z.

10.6.2 Equivalent Contours

Cauchy's integral theorem enables us to replace an integral about an arbitrary simply closed contour by an integral about a more conveniently shaped region, often a circle. Consider the integral $\oint_{C_1} f(z)\,dz$, where the contour C_1 is portrayed in Fig. 10.13. We call C_2 an equivalent contour to C_1 if

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz \tag{10.6.15}$$

This raises the question: Under what circumstances are C_1 and C_2 equivalent contours?

Suppose that $f(z)$ is analytic in the region bounded by C_1 and C_2 and on these contours, as in Fig. 10.13. Then introduce the line segment C_3 joining C_1 to C_2. Let C be the contour made up of C_1 (counterclockwise), C_3 to C_2, C_2 (clockwise), and C_3 from C_2 to C_1. By Cauchy's integral theorem,

$$\oint_C f(z)\,dz = 0 \tag{10.6.16}$$

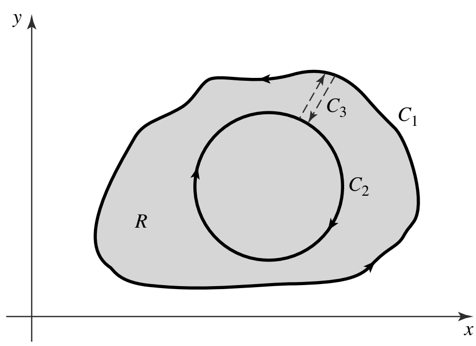


Figure 10.13 Equivalent contours C_1 and C_2.

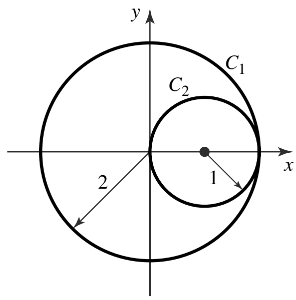
However, by construction of C

$$\oint_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_3} f(z)\,dz - \int_{C_2} f(z)\,dz - \int_{C_3} f(z)\,dz \qquad (10.6.17)$$

So, using Eq. 10.6.16 in Eq. 10.6.17, we see that C_2 is equivalent to C_1 since the two integrals on C_3 cancel.

EXAMPLE 10.6.2

Evaluate the integral $\oint f(z)\,dz$ around the circle of radius 2 with center at the origin if $f(z) = 1/(z-1)$.



▶ Solution

The given function $f(z)$ is not analytic at $z = 1$, a point in the interior domain defined by the contour C_1. However, $f(z)$ is analytic between and on the two circles. Hence,

$$\oint_{C_1} \frac{dz}{z-1} = \oint_{C_2} \frac{dz}{z-1}$$

The contour C_2 is a unit circle centered at $z = 1$. For this circle we have

$$z - 1 = e^{i\theta} \quad \text{and} \quad dz = ie^{i\theta}\,d\theta$$

where θ is now measured with respect to a radius emanating from $z = 1$. The integral becomes

$$\oint_{C_1} \frac{dz}{z-1} = \oint_{C_2} \frac{dz}{z-1} = \int_0^{2\pi} \frac{ie^{i\theta}\,d\theta}{e^{i\theta}} = 2\pi i$$

Observe that this integration is independent of the radius of the circle with center at $z = 1$; a circle of any radius would serve our purpose. Often we choose a circle of radius ϵ, a very small radius. Also, note that the integration around any curve enclosing the point $z = 1$, whether it is a circle or not, would give the value $2\pi i$.

Problems

Evaluate $\oint f(z)\,dz$ for each function, where the path of integration is the unit circle with center at the origin.

1. e^z
2. $\sin z$
3. $1/z^3$
4. $\dfrac{1}{z-2}$
5. $1/\bar{z}$
6. $\dfrac{1}{z^2-5z+6}$

Evaluate $\oint f(z)\,dz$ by direct integration using each function, when the path of integration is the circle with radius 4, center at the origin.

7. $1/z$
8. $\dfrac{1}{z^2-5z+6}$
9. $\dfrac{1}{z-1}$
10. $\dfrac{1}{z^2-4}$
11. z^2+1/z^2
12. $\dfrac{z}{z-1}$

10.7 CAUCHY'S INTEGRAL FORMULAS

We now apply the results of Section 10.6 to the integral

$$\oint_C \frac{f(z)}{z-z_0}\,dz \tag{10.7.1}$$

We suppose that C is a simply closed contour defining the domain D as its interior. The point z_0 is in D and $f(z)$ is assumed analytic throughout D and on C. Figure 10.14 displays this situation. Since the integrand in Eq. 10.7.1 has a singular point at z_0, Cauchy's integral theorem is not directly applicable. However, as we have seen in Section 10.6,

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_{\text{circle}} \frac{f(z)}{z-z_0}\,dz \tag{10.7.2}$$

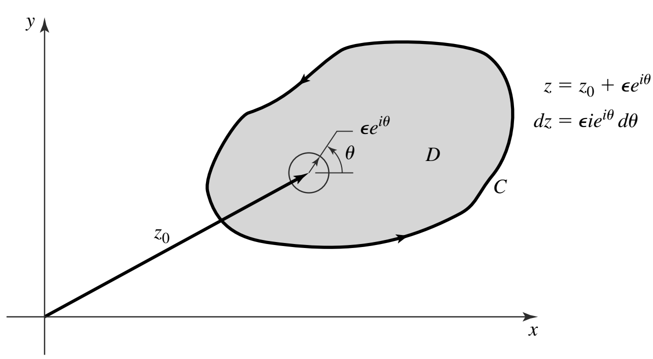


Figure 10.14 Small-circle equivalent to the curve C.

where the circle is as shown in Fig. 10.14. The parameterization of the small circle with radius ϵ leads to

$$\begin{aligned}\oint_{circle} \frac{f(z)}{z-z_0}\,dz &= \int_0^{2\pi} \frac{f(z_0+\epsilon e^{i\theta})}{\epsilon e^{i\theta}}\,\epsilon e^{i\theta}\, i\,d\theta \\ &= i\int_0^{2\pi} f(z_0+\epsilon e^{i\theta})\,d\theta \end{aligned} \tag{10.7.3}$$

Hence, using this equation in Eq. 10.7.2, we find

$$\oint_C \frac{f(z)}{z-z_0}\,dz = i\int_0^{2\pi} f(z_0+\epsilon e^{i\theta})\,d\theta \tag{10.7.4}$$

Now, as $\epsilon \to 0$, we have $f(z_0+\epsilon e^{i\theta}) \to f(z_0)$, which suggests that

$$i\int_0^{2\pi} f(z_0+\epsilon e^{i\theta})\,d\theta = i\int_0^{2\pi} f(z_0)\,d\theta = i f(z_0)\,2\pi \tag{10.7.5}$$

Hence, we conjecture that

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0) \tag{10.7.6}$$

This is *Cauchy's integral formula,* usually written as

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz \tag{10.7.7}$$

We prove Eq. 10.7.5, and thereby Eq. 10.7.7, by examining

$$\left|\int_0^{2\pi} [f(z_0+\epsilon e^{i\theta}) - f(z_0)]\,d\theta\right| \le M2\pi \tag{10.7.8}$$

by Theorem 10.2. However,

$$M = \max|f(z_0+\epsilon e^{i\theta}) - f(z_0)| \tag{10.7.9}$$

around the small circle $z - z_0 = \epsilon e^{i\theta}$. Since $f(z)$ is analytic, it is continuous and so $M \to 0$ as $\epsilon \to 0$. Therefore, inequality 10.7.8 actually implies the equality

$$\int_0^{2\pi} [f(z_0+\epsilon e^{i\theta}) - f(z_0)]\,d\theta = 0 \tag{10.7.10}$$

and Eq. 10.7.5 is proved.

We can obtain an expression for the derivative of $f(z)$ at z_0 by using Cauchy's integral formula in the definition of a derivative as follows:

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z_0 \to 0} \frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z_0} \\ &= \lim_{\Delta z_0 \to 0} \frac{1}{\Delta z_0}\left[\frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z - z_0 - \Delta z_0} - \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz\right] \\ &= \lim_{\Delta z_0 \to 0} \frac{1}{\Delta z_0}\left[\frac{1}{2\pi i}\oint_C f(z)\left(\frac{1}{z - z_0 - \Delta z_0} - \frac{1}{z - z_0}\right)dz\right] \\ &= \lim_{\Delta z_0 \to 0} \frac{1}{\Delta z_0}\left[\frac{\Delta z_0}{2\pi i}\oint_C \frac{f(z)\,dz}{(z - z_0 - \Delta z_0)(z - z_0)}\right. \\ &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^2}\,dz \end{aligned} \tag{10.7.11}$$

(This last equality needs to be proved in a manner similar to the proof of Cauchy's integral formula, 10.7.7.) In a like manner we can show that

$$f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^3}\,dz \tag{10.7.12}$$

or, in general

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz \tag{10.7.13}$$

We often refer to the family of formulas in Eq. 10.7.13 as *Cauchy's integral formulas.*

Cauchy's integral formula 10.7.7 allows us to determine the value of an analytic function at any point z_0 interior to a simply connected region by integrating around a curve C surrounding the region. Only values of the function on the boundary are used. Thus, we note that if an analytic function is prescribed on the entire boundary of a simply connected region, the function and all its derivatives can be determined at all interior points. We can write Eq. 10.7.7 in the alternative form

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w - z}\,dw \tag{10.7.14}$$

where z is any interior point such as that shown in Fig. 10.15. The complex variable w is simply a dummy variable of integration that disappears in the integration process. Cauchy's integral formula is often used in this form.

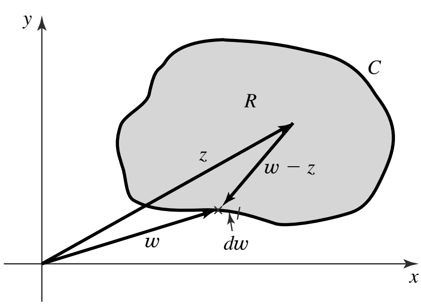


Figure 10.15 Integration variables for Cauchy's integral theorem.

EXAMPLE 10.7.1

Find the value of the integral $\oint z^2/(z^2-1)\,dz$ around the unit circle with center at (a) $z=1$, (b) $z=-1$, and (c) $z=\frac{1}{2}$.

► Solution

Using Cauchy's integral formula (Eq. 10.7.7), we must make sure that $f(z)$ is analytic in the unit circle, and that z_0 lies within the circle.
(a) With the center of the unit circle at $z=1$, we write

$$\oint \frac{z^2}{z^2-1}\,dz = \oint \frac{z^2/(z+1)}{z-1}\,dz$$

where we recognize that

$$f(z) = \frac{z^2}{z+1}$$

This function in analytic at $z=1$ and in the unit circle. Hence, at that point

$$f(1) = \tfrac{1}{2}$$

and we have

$$\oint \frac{z^2}{z^2-1}\,dz = 2\pi i(\tfrac{1}{2}) = \pi i$$

(b) with the center of the unit circle at $z=-1$, we write

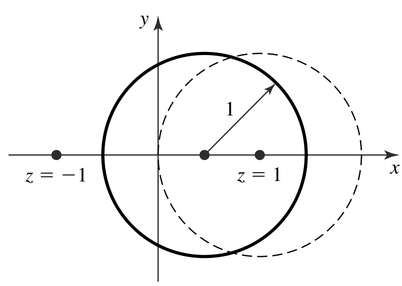
$$\oint \frac{z^2}{z^2-1}\,dz = \oint \frac{z^2/(z-1)}{z+1}\,dz$$

where

$$f(z) = \frac{z^2}{z-1} \quad \text{and} \quad f(-1) = -\tfrac{1}{2}$$

There results

$$\oint \frac{z^2}{z^2-1}\,dz = 2\pi i(-\tfrac{1}{2}) = -\pi i$$



EXAMPLE 10.7.1 *(Continued)*

(c) Rather than integrating around the unit circle with center at $z = \frac{1}{2}$, we can integrate around any curve enclosing the point $z = 1$ just so the curve does not enclose the other singular point at $z = -1$. Obviously, the unit circle of part (a) is an acceptable alternative curve. Hence,

$$\oint \frac{z^2}{z^2 - 1}\, dz = \pi i$$

EXAMPLE 10.7.2

Evaluate the integrals

$$\oint \frac{z^2 + 1}{(z - 1)^2}\, dz \quad \text{and} \quad \oint \frac{\cos z}{z^3}\, dz$$

around the circle $|z| = 2$.

▶ Solution

Using Eq. 10.7.11, we can write the first integral as

$$\oint \frac{z^2 + 1}{(z - 1)^2}\, dz = 2\pi i f'(1)$$

where

$$f(z) = z^2 + 1 \quad \text{and} \quad f'(z) = 2z$$

Then

$$f'(1) = 2$$

The value of the integral is then determined to be

$$\oint \frac{z^2 + 1}{(z - 1)^2}\, dz = 2\pi i(2) = 4\pi i$$

For the second integral of the example, we have

$$\oint \frac{\cos z}{z^3}\, dz = \frac{2\pi i}{2!} f''(0)$$

where

$$f(z) = \cos z \quad \text{and} \quad f''(z) = -\cos z$$

At the origin

$$f''(0) = -1$$

The integral is then

$$\oint \frac{\cos z}{z^3} = \frac{2\pi i}{2!}(-1) = -\pi i$$

Problems

Find the value of each integral around the circle $|z| = 2$ using Cauchy's integral formula.

1. $\oint \frac{\sin z}{z}\, dz$
2. $\oint \frac{e^z}{z-1}\, dz$
3. $\oint \frac{z}{z^2+4z+3}\, dz$
4. $\oint \frac{z^2-1}{z^3-z^2+9z-9}\, dz$
5. $\oint \frac{\cos z}{z-1}\, dz$
6. $\oint \frac{z^2}{z+i}\, dz$

Evaluate the integral $\oint (z-1)/(z^2+1)\, dz$ around each curve.

7. $|z-i| = 1$
8. $|z+i| = 1$
9. $|z| = 1/2$
10. $|z-1| = 1$
11. $|z| = 2$
12. The ellipse $2x^2 + (y+1)^2 = 1$

If the curve C is the circle $|z| = 2$, determine each integral.

13. $\oint_C \frac{\sin z}{z^2}\, dz$
14. $\oint_C \frac{z-1}{(z+1)^2}\, dz$
15. $\oint_C \frac{z^2}{(z-1)^3}\, dz$
16. $\oint_C \frac{\cos z}{(z-1)^2}\, dz$
17. $\oint_C \frac{e^z}{(z-i)^2}\, dz$
18. $\oint \frac{\sinh z}{z^4}\, dz$

10.8 TAYLOR SERIES

The representation of an analytic function by an infinite series is basic in the application of complex variables. Let us show that if $f(z)$ is analytic at the point $z = a$, then $f(z)$ can be represented by a series of powers of $z - a$. Expand the quantity $(w-z)^{-1}$ as follows:

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\left[\frac{1}{1-\dfrac{z-a}{w-a}}\right]$$

$$= \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \cdots + \left(\frac{z-a}{w-a}\right)^{n-1} + R_n(z, w)\right] \tag{10.8.1}$$

where

$$R_n(z, w) = \frac{1}{1-\dfrac{z-a}{w-a}}\left(\frac{z-a}{w-a}\right)^n \tag{10.8.2}$$

Equation 10.8.1 is the algebraic identity

$$\frac{1}{1-r} = 1 + r + r^2 + \cdots + r^{n-1} + \frac{r^n}{1-r} \tag{10.8.3}$$

We can now substitute the expansion of $1/(w-z)$ in Cauchy's integral formula 10.7.14, to obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-a} \left[1 + \frac{z-a}{w-a} + \cdots + \left(\frac{z-a}{w-a} \right)^{n-1} + R_n(z,w) \right] dw \\ &= \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-a} dw + \frac{z-a}{2\pi i} \oint_C \frac{f(w)}{(w-a)^2} dw + \cdots \\ &\quad + \frac{(z-a)^{n-1}}{2\pi i} \oint_C \frac{f(w)}{(w-a)^n} dw + \frac{(z-a)^n}{2\pi i} \oint_C \frac{f(w)}{(w-z)(w-a)^n} dw \end{aligned} \tag{10.8.4}$$

where we have simplified $R_n(z, w)$ by noting that Eq. 10.8.2 is equivalently

$$R_n(z,w) = \frac{w-a}{w-z} \left(\frac{z-a}{w-a} \right)^n = \frac{(z-a)^n}{(w-z)(w-a)^{n-1}} \tag{10.8.5}$$

Now, the Cauchy integral formulas 10.7.13 may be used in Eq. 10.8.4 to write

$$\begin{aligned} f(z) =& f(a) + \frac{1}{1!} f'(a)(z-a) + \cdots + \frac{1}{(n-1)!} f^{(n-1)}(a)(z-a)^{n-1} \\ &+ \frac{(z-a)^n}{2\pi i} \oint_C \frac{f(w)\,dw}{(w-z)(w-a)^n} \end{aligned} \tag{10.8.6}$$

The integral in this equation is the *remainder term* and for appropriate z, w, and a, this term tends to zero as n tends to infinity. Suppose that C is a circle centered at $z = a$ and z is a point inside the circle C (see Fig. 10.16). We assume that $f(z)$ is analytic on C and in its interior, that

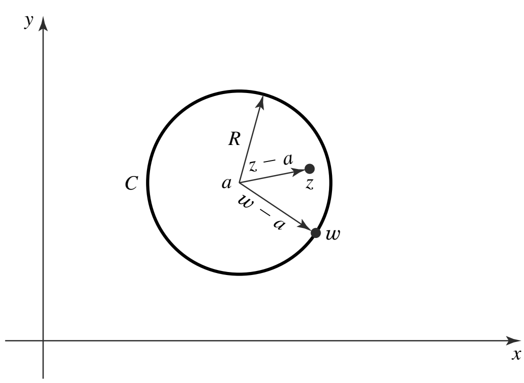


Figure 10.16 Circular region of convergence for the Taylor series.

$|z-a| = r$, and hence that $r < R = |w-a|$. Let $M = \max|f(z)|$ for z on C. Then by Theorem 10.2, and the fact that $|w-z| \geq R - r$ (see Fig. 10.16),

$$\left|\frac{(z-a)^n}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-z)(w-a)^n}\right| \leq \frac{r^n}{2\pi}\frac{M}{R-r}\frac{2\pi R}{R^n} = M\frac{R}{R-r}\left(\frac{r}{R}\right)^n \tag{10.8.7}$$

Since $r < R$, $(r/R)^n \to 0$ *as* $n \to \infty$. We have thus established the convergence of the famous Taylor series

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{z-a}{2!} + \cdots + f^{(n)}(a)\frac{(z-a)^n}{n!} + \cdots \tag{10.8.8}$$

The convergence holds in the largest circle about $z = a$ in which $f(z)$ is analytic. If $|z-a| = R$ is this circle, then R is the *radius of convergence* and we are assured of the convergence in the open set $|z-a| < R$. We mention in passing that the series 10.8.8 must diverge for those z, $|z-a| > R$. The convergence or divergence on the circle $|z-a| = R$ is a more difficult issue which we choose not to explore; suffice it to say that $f(z)$ must have a singular point somewhere on $|z-a| = R$ by definition of R.

The discussion above applies equally well to a circular region about the origin, $a = 0$. The resultant series expression

$$f(z) = f(0) + f'(0)z + f''(0)\frac{z^2}{2!} + \cdots \tag{10.8.9}$$

is sometimes called a *Maclaurin series,* especially if $z = x$ is real.

EXAMPLE 10.8.1

Use the Taylor series representation of $f(z)$ and find a series expansion about the origin for (a) $f(z) = \sin z$, (b) $f(z) = e^z$, and (c) $f(z) = 1/(1-z)^m$.

► Solution

(a) To use Eq. 10.8.9, we must evaluate the derivatives at $z = 0$. They are

$$\begin{aligned} f'(0) &= \cos 0 = 1, \quad f''(0) = -\sin 0 = 0, \\ f'''(0) &= -\cos 0 = -1, \quad \text{etc.} \end{aligned}$$

The Taylor series is then, with $f(z) = \sin z$,

$$\begin{aligned} \sin z &= \sin 0 + 1\cdot(z-0) + 0\cdot\frac{(z-0)^2}{2!} - 1\cdot\frac{(z-0)^3}{3!} + \cdots \\ &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \end{aligned}$$

This series is valid for all z since no singular point exists in the xy plane. It is the series given in Section 10.3 to define $\sin z$.

EXAMPLE 10.8.1 (*Continued*)

(b) For the second function the derivatives are

$$f'(0) = e^0 = 1, \qquad f''(0) = e^0 = 1, \qquad f'''(0) = e^0 = 1, \ldots$$

The Taylor series for $f(z) = e^z$ is then

$$\begin{aligned} e^z &= e^0 + 1 \cdot z + 1 \cdot \frac{z^2}{2!} + 1 \cdot \frac{z^3}{3!} + \cdots \\ &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \end{aligned}$$

This series is valid for all z. Note that this is precisely the series we used in Section 10.3 to define e^z.
(c) We determine the derivatives to be

$$\begin{aligned} f'(z) &= m(1-z)^{-m-1}, \quad f''(z) = m(m+1)(1-z)^{-m-2}, \\ f'''(z) &= m(m+1)(m+2)(1-z)^{-m-3}, \ldots \end{aligned}$$

Substitute into Taylor series to obtain

$$\begin{aligned} \frac{1}{(1-z)^m} &= \frac{1}{(1-0)^m} + m(1-0)^{-m-1}z + m(m+1)(1-0)^{-m-2}\frac{z^2}{2!} + \cdots \\ &= 1 + mz + m(m+1)\frac{z^2}{2!} + m(m+1)(m+2)\frac{z^3}{3!} + \cdots \end{aligned}$$

This series converges for $|z| < 1$ and does not converge for $|z| \geq 1$ since a singular point exists at $z = 1$. Using $m = 1$, the often used expression for $1/(1-z)$ results,

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

EXAMPLE 10.8.2

Find the Taylor series representation of $\ln(1+z)$ by noting that

$$\frac{d}{dz}\ln(1+z) = \frac{1}{1+z}$$

► Solution

First, let us write the Taylor series expansion of $1/(1+z)$. It is, using the results of Example 10.8.1(c)

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = 1 - z + z^2 - z^3 + \cdots$$

EXAMPLE 10.8.2 *(Continued)*

Now, we can perform the integration

$$\int d[\ln(1+z)]\,dz = \int \frac{1}{1+z}\,dz$$

using the series expansion of $1/(1+z)$ to obtain

$$\ln(1+z) = \int \frac{1}{1+z}\,dz = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots + C$$

The constant of integration $C = 0$, since when $z = 0$, $\ln(1) = 0$. The power-series expansion is finally

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{2} - \cdots$$

This series is valid for $|z| < 1$ since a singularity exists at $z = -1$.

EXAMPLE 10.8.3

Determine the Taylor series expansion of

$$f(z) = \frac{1}{(z^2 - 3z + 2)}$$

about the origin.

► Solution

First, represent the function $f(z)$ as partial fractions; that is,

$$\frac{1}{z^2 - 3z + 2} = \frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1}$$

The series representations are then, using the results of Example 10.8.1(c),

$$\begin{aligned}
\frac{1}{z-1} &= -\frac{1}{1-z} = -(1 + z + z^2 + \cdots) \\
\frac{1}{z-2} &= -\frac{1}{2}\left(\frac{1}{1-z/2}\right) = -\frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right] \\
&= -\frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right]
\end{aligned}$$

Finally, the difference of the two series is

$$\frac{1}{z^2 - 3z - 2} = \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \frac{15}{16}z^3 + \cdots$$

We could also have multiplied the two series together to obtain the same result.

EXAMPLE 10.8.4

Find the Taylor series expansion of

$$f(z) = \frac{1}{z^2 - 9}$$

by expanding about the point $z = 1$.

► Solution

We write the function $f(z)$ in partial fractions as

$$\begin{aligned}
\frac{1}{z^2-9} = \frac{1}{(z-3)(z+3)} &= \frac{1}{2}\left(\frac{1}{z-3}\right) - \frac{1}{6}\left(\frac{1}{z+3}\right) \\
&= -\frac{1}{6}\left[\frac{1}{2-(z-1)}\right] - \frac{1}{6}\left[\frac{1}{4+(z-1)}\right] \\
&= -\frac{1}{12}\left[\frac{1}{1-\dfrac{z-1}{2}}\right] - \frac{1}{24}\left[\frac{1}{1-\left(-\dfrac{z-1}{4}\right)}\right]
\end{aligned}$$

Now, we can expand in a Taylor series as

$$\begin{aligned}
\frac{1}{z^2-9} &= -\frac{1}{12}\left[1 + \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 + \left(\frac{z-1}{2}\right)^3 + \cdots\right] \\
&\quad - \frac{1}{24}\left[1 - \frac{z-1}{4} + \left(\frac{z-1}{4}\right)^2 - \left(\frac{z-1}{4}\right)^3 + \cdots\right] \\
&= -\frac{1}{8} - \frac{1}{32}(z-1) - \frac{3}{128}(z-1)^2 - \frac{5}{512}(z-1)^3 + \cdots
\end{aligned}$$

The nearest singularity is at the point $z = 3$; hence, the radius of convergence is 2; that is $|z-1| < 2$. This is also obtained from the first ratio since $|(z-1)/2| < 1$ or $|z-1| < 2$. The second ratio is convergent if $|-(z-1)/4| < 1$ or $|z-1| < 4$; thus, it is the first ratio that limits the radius of convergence.

10.8.1 *Maple* Applications

Maple's `taylor` command can compute the Taylor series of a function. As we described earlier with *Maple*'s `diff` command, the `taylor` command is valid with real or complex variables. There are three arguments for the `taylor` command: the function, the point of expansion, and the number of terms that is desired. The output must be interpreted with care. For example, to get the first four terms of the Taylor series representation of sin z about the origin:

```
>taylor(sin(z), z=0, 4);
```

$$z - \frac{1}{6}z^3 + O(z^4)$$

Here, the first four terms are 0, z, 0, and $-z^3/6$. The $O(z^4)$ indicates that the rest of the terms have power z^4 or higher. For e^z and $1/(1-z)^m$:

```
>taylor(exp(z), z=0, 4);
```

$$1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + O(z^4)$$

```
>taylor(1/(1-z)^m, z=0, 4):
>simplify(%);
```

$$1 + m\,z + \left(\frac{1}{2}m^2 + \frac{1}{2}m\right)z^2 + \left(\frac{1}{6}m^3 + \frac{1}{2}m^2 + \frac{1}{3}m\right)z^3 + O(z^4)$$

Expanding around a complex number can also create Taylor series. For instance, expanding e^z about $z = i\pi$ yields

```
>taylor(exp(z), z=I*Pi, 4);
```

$$-1 - (z - \pi I) - \frac{1}{2}(z - \pi I)^2 - \frac{1}{6}(z - \pi I)^3 + O((z - \pi I)^4)$$

Problems

Using the Taylor series, find the expansion about the origin for each function. State the radius of convergence.

1. $\cos z$

2. $\dfrac{1}{1+z}$

3. $\ln(1+z)$

4. $\dfrac{z-1}{z+1}$

5. $\cosh z$

6. $\sinh z$

For the function $1/(z-2)$, determine the Taylor series expansion about each of the given points. Use the known series expansion for $1/(1-z)$. State the radius of covergence for each.

7. $z = 0$

8. $z = 1$

9. $z = i$

10. $z = -1$

11. $z = 3$

12. $z = -2i$

Using known series expansions, find the Taylor series expansion about the origin of each of the following.

13. $\dfrac{1}{1-z^2}$

14. $\dfrac{z-1}{1+z^3}$

15. $\dfrac{z^2+3}{2-z}$

16. $\dfrac{1}{z^2-3z-4}$

17. e^{-z^2}

18. e^{2-z}

19. $\sin \pi z$

20. $\sin z^2$

21. $\dfrac{\sin z}{1-z}$

22. $e^z \cos z$

23. $\tan z$

24. $\dfrac{\sin z}{e^{-z}}$

What is the Taylor series expansion about the origin for each of the following?

25. $\int_0^z e^{-w^2}\,dw$

26. $\int_0^z \sin w^2\,dw$

27. $\int_0^z \frac{\sin w}{w}\,dw$

28. $\int_0^z \cos w^2\,dw$

29. Find the Taylor series expansion about the origin of $f(z) = \tan^{-1} z$ by recognizing that $f'(z) = 1/(1+z^2)$.

Determine the Taylor series expansion of each function about the point $z = a$.

30. e^z, $a = 1$

31. $\frac{1}{1-z}$, $a = 2$

32. $\sin z$, $a = \frac{\pi}{2}$

33. $\ln z$, $a = 1$

34. $\frac{1}{z^2 - z - 2}$, $a = 0$

35. $\frac{1}{z^2}$, $a = 1$

Use *Maple* to solve

36. Problem 1

37. Problem 2

38. Problem 3

39. Problem 4

40. Problem 5

41. Problem 6

42. Problem 7

43. Problem 8

44. Problem 9

45. Problem 10

46. Problem 11

47. Problem 12

48. Problem 25

49. Problem 26

50. Problem 27

51. Problem 28

52. Problem 29

53. Problem 30

54. Problem 31

55. Problem 32

56. Problem 33

57. Problem 34

58. Problem 35

10.9 LAURENT SERIES

There are many applications in which we wish to expand a function $f(z)$ in a series about a point $z = a$, which is a singular point. Consider the annulus shown in Fig. 10.17a. The function $f(z)$ is analytic in the annular region; however, there may be singular points inside the smaller circle or outside the larger circle. The possibility of a singular point inside the smaller circle bars us from expanding in a Taylor series, since the function $f(z)$ must be analytic at all interior points. We can apply Cauchy's integral formula to the multiply connected region by cutting the region as shown in Fig. 10.17b, thereby forming a simply connected region bounded by the curve C'. Cauchy's integral formula is then

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{C'} \frac{f(w)}{w-z}\,dw \\ &= \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z}\,dw \end{aligned} \tag{10.9.1}$$

where C_1 and C_2 are both traversed in the counterclockwise direction. The negative sign results because the direction of integration was reversed on C_1. Now, let us express the quantity

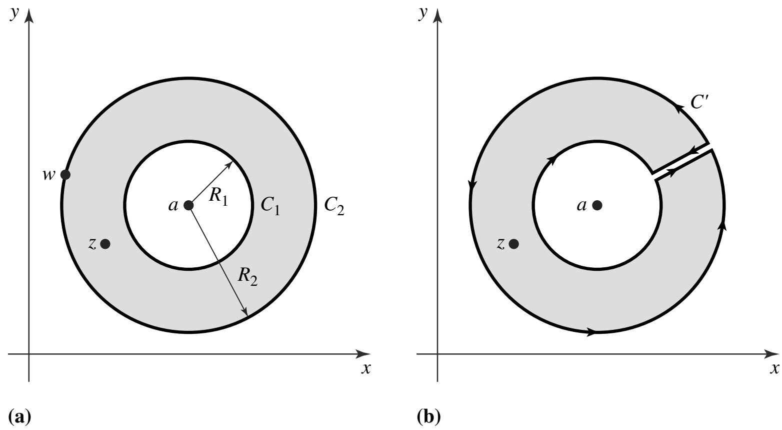


Figure 10.17 Annular region inside of which a singular point exists.

$(w - z)^{-1}$ in the integrand of Eq. 10.9.1 in a form that results in positive powers of $(z - a)$ in the C_2 integration and that results in negative powers in the C_1 integration. If no singular points exist inside C_1, then the coefficients of the negative powers will all be zero and a Taylor series will result. Doing this, we have

$$\begin{aligned} f(z) = &\frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{w - a} \left[\frac{1}{1 - \dfrac{z - a}{w - a}} \right] dw \\ &+ \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{z - a} \left[\frac{1}{1 - \dfrac{w - a}{z - a}} \right] dw \end{aligned} \tag{10.9.2}$$

By using arguments analogous to those used to prove the convergence of the Taylor series, we can show that Eq. 10.9.2 leads to

$$\begin{aligned} f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \cdots \\ + b_1(z - a)^{-1} + b_2(z - a)^{-2} + \cdots \end{aligned} \tag{10.9.3}$$

where

$$a_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{(w - a)^{n+1}}\, dw, \qquad b_n = \frac{1}{2\pi i} \oint_{C_1} f(w)(w - a)^{n-1}\, dw \tag{10.9.4}$$

The series expression 10.9.3 is a *Laurent series.* The integral expression for the coefficients a_n resembles the formulas for the derivatives of $f(z)$; but this is only superficial, for $f(z)$ may not be defined at $z = a$ and certainly $f(z)$ may not be analytic there. Note, however, that if $f(z)$ is analytic in the circle C_1, the integrand in the integral for b_n is everywhere analytic, requiring the b_n's to all be zero, a direct application of Cauchy's integral theorem. In this case the Laurent series reduces to a Taylor series.

The integral expressions 10.9.4 for the coefficients in the Laurent series are not normally used to find the coefficients. It is known that the series expansion is unique; hence, elementary techniques are usually used to find the Laurent series. This will be illustrated with an example. The region of convergence may be found, in most cases, by putting the desired $f(z)$ in the form $1/(1-z^*)$ so that $|z^*| < 1$ establishes the region of convergence.

Since $|z^*| < 1$, we have the geometric series:

$$\frac{1}{1-z^*} = 1 + z^* + (z^*)^2 + (z^*)^3 + \cdots \tag{10.9.5}$$

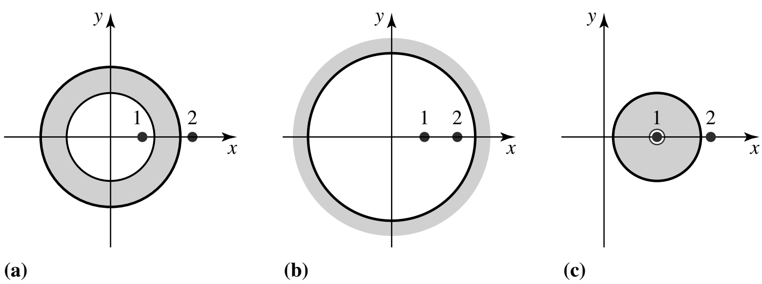
which is a Laurent series expanded about $z^* = 0$.

EXAMPLE 10.9.1

What is the Laurent series expansion of

$$f(z) = \frac{1}{z^2 - 3z + 2}$$

valid in each of the shaded regions shown?



► Solution

(a) To obtain a Laurent series expansion in the shaded region of (a), we expand about the origin. We express the ratio in partial fractions as

$$\begin{aligned}\frac{1}{z^2-3z+2} = \frac{1}{(z-2)(z-1)} &= \frac{1}{z-2} - \frac{1}{z-1}\\ &= -\frac{1}{2}\left(\frac{1}{1-z/2}\right) - \frac{1}{z}\left(\frac{1}{1-1/z}\right)\end{aligned}$$

The first fraction has a singularity at $z/2 = 1$ and can be expanded in a Taylor series that converges if $|z/2| < 1$ or $|z| < 2$. The second fraction has a singularity at $1/z = 1$ and can be expanded in a Laurent series

EXAMPLE 10.9.1 *(Continued)*

that converges if $|1/z| < 1$ or $|z| > 1$. The two fractions are expressed in the appropriate series as

$$-\frac{1}{2}\left(\frac{1}{1-z/2}\right) = -\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$
$$= -\frac{1}{2}-\frac{z}{4}-\frac{z^2}{8}-\frac{z^3}{16}-\cdots$$
$$-\frac{1}{z}\left(\frac{1}{1-1/z}\right) = -\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right]$$
$$= -\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}-\cdots$$

where the first series is valid for $|z| < 2$ and the second series for $|z| > 1$. Adding the two preceding expressions yields the Laurent series

$$\frac{1}{z^2-3z+2} = \cdots-\frac{1}{z^3}-\frac{1}{z^2}-\frac{1}{z}-\frac{1}{2}-\frac{z}{4}-\frac{z^2}{8}-\frac{z^3}{16}-\cdots$$

which is valid in the region $1 < |z| < 2$.
(b) In the region exterior to the circle $|z| = 2$, we expand $1/(z-1)$, as before,

$$\frac{1}{z-1} = \frac{1}{z}\left(\frac{1}{1-1/z}\right) = \frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots$$

which is valid if $|1/z| < 1$ or $|z| > 1$. Now, though, we write

$$\frac{1}{z-2} = \frac{1}{z}\left(\frac{1}{1-2/z}\right) = \frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right]$$
$$= \frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\frac{8}{z^4}+\cdots$$

which is valid if $|2/z| < 1$ or $|z| > 2$. The two preceding series expansions are thus valid for $|z| > 2$, and we have the Laurent series

$$\frac{1}{z^2-3z+2} = \frac{1}{z^2}+\frac{3}{z^3}+\frac{7}{z^4}+\frac{15}{z^5}+\cdots$$

valid in the region $|z| > 2$.
(c) To obtain a series expansion in the region $0 < |z-1| < 1$, we expand about the point $z = 1$ and obtain

$$\frac{1}{z^2-3z+2} = \frac{1}{z-1}\left(-\frac{1}{2-z}\right) = \frac{1}{z-1}\left[\frac{-1}{1-(z-1)}\right]$$
$$= \frac{-1}{z-1}[1+(z-1)+(z-1)^2+(z-1)^3+\cdots]$$
$$= -\frac{1}{z-1}-1-(z-1)-(z-1)^2+\cdots$$

This Laurent series is valid if $0 < |z-1| < 1$.

10.9.1 *Maple* Applications

Laurent series can be computed with *Maple* using the `laurent` command in the `numapprox` package, but only in the case where we are obtaining a series expansion about a specific point. For instance, part (c) of Example 10.9.1 can be calculated in this way:

```
>with(numapprox):
>laurent(1/(z^2-3*z+2), z=1);
```

$$-(z-1)^{-1}-1-(z-1)-(z-1)^2-(z-1)^3-(z-1)^4+O((z-1)^5)$$

This command will not determine the region of validity.

For Example 10.9.1(a), *Maple* would be best used to determine the partial fraction decomposition (using the `convert` command), create the series for each part (using `taylor` to get the correct geometric series—`laurent` applied to $1/(1-1/z)$ will not work—and then combine the series (again using the `convert` command to first strip the big-O terms from the series):

```
>L1 :=laurent(1/(1-z/2); z=0);
```

$$L1 := 1+\frac{1}{2}z+\frac{1}{4}z^2+\frac{1}{8}z^3+\frac{1}{16}z^4+\frac{1}{32}z^5+O(z^6)$$

```
>taylor(1/(1-x), x=0);
```

$$1+x+x^2+x^3+x^4+x^5+O(x^6)$$

```
>L2 :=subs(x=1/z, %);
```

$$L2 := 1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\frac{1}{z^5}+O\left(\frac{1}{z^6}\right)$$

```
>PL1 :=convert(L1,polynom); PL2 :=convert(L2,polynom);
```

$$PL1 := 1+\frac{1}{2}z+\frac{1}{4}z^2+\frac{1}{8}z^3+\frac{1}{16}z^4+\frac{1}{32}z^5$$
$$PL2 := 1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\frac{1}{z^5}$$

```
>-1/2*PL1-1/z*PL2;
```

$$-\frac{1}{2}-\frac{z}{4}-\frac{z^2}{8}-\frac{z^3}{16}-\frac{z^4}{32}-\frac{z^4}{64}-\frac{1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\frac{1}{z^5}}{z}$$

To simplify this expression, we ask *Maple* to `collect` like terms:

```
>collect(-1/2*PL1-1/z*PL2, z);
```

$$-\frac{1}{2}-\frac{z^5}{64}-\frac{z^4}{32}-\frac{z^3}{16}-\frac{z^2}{8}-\frac{z}{4}-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}-\frac{1}{z^5}-\frac{1}{z^6}$$

Example 10.9.1(b) can be done in the same way.

Problems

Expand each function in a Laurent series about the origin, convergent in the region $0 < |z| < R$. State the radius of convergence R.

1. $\frac{1}{z^2}\sin z$

2. $\frac{1}{z^2 - 2z}$

3. $\frac{1}{z(z^2 + 3z + 2)}$

4. $\frac{e^{z-1}}{z}$

For each function, find all Taylor series and Laurent series expansions about the point $z = a$ and state the region of convergence for each.

5. $\frac{1}{z}, \quad a = 1$

6. $e^{1/z}, \quad a = 0$

7. $\frac{1}{1-z}, \quad a = 0$

8. $\frac{1}{1-z}, \quad a = 1$

9. $\frac{1}{1-z}, \quad a = 2$

10. $\frac{1}{z(z-1)}, \quad a = 0$

11. $\frac{z}{1-z^2}, \quad a = 1$

12. $\frac{1}{z^2+1}, \quad a = i$

13. $\frac{1}{(z+1)(z-2)}, \quad a = 0$

14. $\frac{1}{(z+1)(z-2)}, \quad a = -1$

15. $\frac{1}{(z+1)(z-2)}, \quad a = 2$

Use *Maple* to solve

16. Problem 1

17. Problem 2

18. Problem 3

19. Problem 4

20. Problem 5

21. Problem 6

22. Problem 7

23. Problem 8

24. Problem 9

25. Problem 10

26. Problem 11

27. Problem 12

28. Problem 13

29. Problem 14

30. Problem 15

31. Determine the Laurent series expansion of

$$f(z) = \frac{3}{z^2 - 6z + 8}$$

which is valid in region given.

(a) $0 < |z-4| < 1$

(b) $|z| > 5$

(c) $2 < |z| < 4$

10.10 RESIDUES

In this section we shall present a technique that is especially useful when evaluating certain types of real integrals. Suppose that a function $f(z)$ is singular at the point $z = a$ and is analytic at all other points within some circle with center at $z = a$. Then $f(z)$ can be expanded in the Laurent series (see Eq. 10.9.3)

$$f(z) = \cdots + \frac{b_m}{(z-a)^m} + \cdots + \frac{b_2}{(z-a)^2} + \frac{b_1}{z-a} + a_0 + a_1(z-a) + \cdots \tag{10.10.1}$$

Three cases arise. First, all the coefficients $b_1, b_2, \ldots$ are zero. Then $f(z)$ is said to have a *removable singularity.* The function $(\sin z)/z$ has a removable singularity at $z = 0$. Second, only a finite number of the b_n are nonzero. Then $f(z)$ has a *pole* at $z = a$. If $f(z)$ has a pole, then

$$f(z) = \frac{b_m}{(z-a)^m} + \cdots + \frac{b_1}{z-a} + a_0 + a_1(z-a) + \cdots \tag{10.10.2}$$

where $b_m \neq 0$. In this case we say that the pole at $z = a$ is of *order m*. Third, if infinitely many b_n are not zero, then $f(z)$ has an *essential singularity* at $z = a$. The function $e^{1/z}$ has the Laurent expansion

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\, z^2} + \cdots + \frac{1}{n!\, z^n} + \cdots \tag{10.10.3}$$

valid for all z, $|z| > 0$. The point $z = 0$ is an essential singularity of $e^{1/z}$. It is interesting to observe that rational fractions have poles or removable singularities as their only singularities.

From the expression 10.9.4 we see that

$$b_1 = \frac{1}{2\pi i} \oint_{C_1} f(w)\, dw \tag{10.10.4}$$

Hence, the integral of a function $f(z)$ about some connected curve surrounding one singular point is given by

$$\oint_{C_1} f(z)\, dz = 2\pi i b_1 \tag{10.10.5}$$

where b_1 is the coefficient of the $(z-a)^{-1}$ term in the Laurent series expansion at the point $z = a$. The quantity b_1 is called the *residue* of $f(z)$ at $z = a$. Thus, to find the integral of a function about a singular point,[11] we simply find the Laurent series expansion and use the relationship 10.10.5. An actual integration is not necessary. If more than one singularity exists within the closed curve *C*, we make it simply connected by cutting it as shown in Fig. 10.18. Then an application of Cauchy's integral theorem gives

$$\oint_C f(z)\, dz + \oint_{C_1} f(z)\, dz + \oint_{C_2} f(z)\, dz + \oint_{C_3} f(z)\, dz = 0 \tag{10.10.6}$$

since $f(z)$ is analytic at all points in the region outside the small circles and inside *C*. If we reverse the direction of integration on the integrals around the circles, there results

$$\oint_C f(z)\, dz = \oint_{C_1} f(z)\, dz + \oint_{C_2} f(z)\, dz + \oint_{C_3} f(z)\, dz \tag{10.10.7}$$

[11]Recall that we only consider isolated singularities.

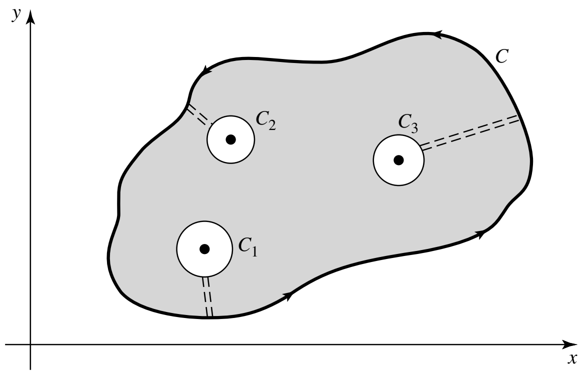


Figure 10.18 Integration about a curve that surrounds singular points.

In terms of the residues at the points, we have Cauchy's *residue theorem,*

$$\oint_C f(z)\, dz = 2\pi i[(b_1)_{a_1} + (b_1)_{a_2} + (b_1)_{a_3}] \tag{10.10.8}$$

where the b_1's are coefficients of the $(z-a)^{-1}$ terms of the Laurent series expansions at each of the points.

Another technique, often used to find the residue at a particular singular point, is to multiply the Laurent series (10.10.2) by $(z-a)^m$, to obtain

$$\begin{aligned}(z-a)^m f(z) &= b_m + b_{m-1}(z-a) + \cdots \\ &\quad + b_1(z-a)^{m-1} + a_0(z-a)^m + a_1(z-a)^{m+1} + \cdots\end{aligned} \tag{10.10.9}$$

Now, if the series above is differentiated $(m-1)$ times and we let $z=a$, the residue results; that is,

$$b_1 = \frac{1}{(m-1)!}\left\{\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]\right\}_{z=a} \tag{10.10.10}$$

Obviously, the order of the pole must be known before this method is useful. If $m = 1$, no differentiation is required and the residue results from

$$\lim_{z\to a}(z-a)f(z)$$

The residue theorem can be used to evaluate certain real integrals. Several examples will be presented here. Consider the real integral

$$I = \int_0^{2\pi} g(\cos\theta, \sin\theta)\, d\theta \tag{10.10.11}$$

where $g(\cos\theta, \sin\theta)$ is a rational[12] function of $\cos\theta$ and $\sin\theta$ with no singularities in the interval $0 \le \theta < 2\pi$. Let us make the substitution

$$e^{i\theta} = z \tag{10.10.12}$$

[12]Recall that a rational function can be expressed as the ratio of two polynomials.

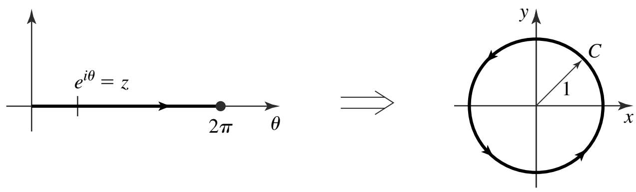


Figure 10.19 Paths of integration.

resulting in

$$\begin{aligned}\cos\theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right)\\ \sin\theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right)\\ d\theta &= \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}\end{aligned} \tag{10.10.13}$$

As θ ranges from, 0 to 2π, the complex variable z moves around the unit circle, as shown in Fig. 10.19, in the counterclockwise sense. The real integral now takes the form

$$I = \oint_C \frac{f(z)}{iz}\, dz \tag{10.10.14}$$

The residue theorem can be applied to the integral above once $f(z)$ is given. All residues inside the unit circle must be accounted for.

A second real integral that can be evaluted using the residue theorem is the integral

$$I = \int_{-\infty}^{\infty} f(x)\, dx \tag{10.10.15}$$

where $f(x)$ is the rational function

$$f(x) = \frac{p(x)}{q(x)} \tag{10.10.16}$$

and $q(x)$ has no real zeros and is of degree at least 2 greater than $p(x)$. Consider the corresponding integral

$$I_1 = \oint_C f(z)\, dz \tag{10.10.17}$$

where C is the closed path shown in Fig. 10.20. If C_1 is the semicircular part of curve C, Eq. 10.10.17 can be written as

$$I_1 = \int_{C_1} f(z)\, dz + \int_{-R}^{R} f(x)\, dx = 2\pi i \sum_{n=1}^{N} (b_1)_n \tag{10.10.18}$$

where Cauchy's residue theorem has been used. In this equation, N represents the number of singularities in the upper half-plane contained within the semicircle. Let us now show that

$$\int_{C_1} f(z)\, dz \to 0 \tag{10.10.19}$$

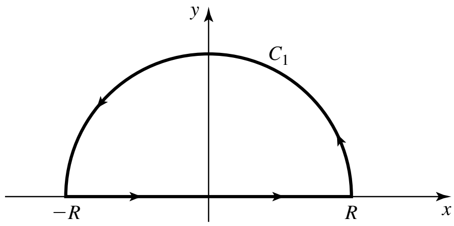


Figure 10.20 Path of integration.

as $R \to \infty$. Using Eq. 10.10.16 and the restriction that $q(z)$ is of degree at least 2 greater than $p(z)$, we have

$$|f(z)| = \left|\frac{p(z)}{q(z)}\right| = \frac{|p(z)|}{|q(z)|} \sim \frac{1}{R^2} \tag{10.10.20}$$

Then there results

$$\left|\int_{C_1} f(x)\, dz\right| \le |f_{\max}|\,\pi R \sim \frac{1}{R} \tag{10.10.21}$$

from Theorem 10.2. As the radius R of the semicircle approaches ∞, we see that

$$\int_{C_1} f(z)\, dz \to 0 \tag{10.10.22}$$

Finally,

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{n=1}^{N} (b_1)_n \tag{10.10.23}$$

where the b_1's include the residues of $f(z)$ at all singularities in the upper half-plane.

A third real integral that may be evaluated using the residue theorem is

$$I = \int_{-\infty}^{\infty} f(x) \sin mx\, dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x) \cos mx\, dx \tag{10.10.24}$$

Consider the complex integral

$$I_1 = \oint_C f(z) e^{imz} dz \tag{10.10.25}$$

where m is positive and C is the curve of Fig. 10.20. If we limit ourselves to the upper half-plane so that $y \ge 0$,

$$|e^{imz}| = |e^{imx}||e^{-my}| = e^{-my} \le 1 \tag{10.10.26}$$

We then have

$$|f(z)e^{imz}| = |f(z)||e^{imz}| \le |f(z)| \tag{10.10.27}$$

The remaining steps follow as in the previous example for $\int_{-\infty}^{\infty} f(z)\,dz$ using Fig. 10.20. This results in

$$\int_{-\infty}^{\infty} f(x)e^{imx}\,dx = 2\pi i \sum_{n=1}^{N} (b_1)_n \tag{10.10.28}$$

where the b_1's include the residues of $[f(z)e^{imz}]$ at all singularities in the upper half-plane. Then the value of the integrals in Eq. 10.10.24 are either the real or imaginary parts of Eq. 10.10.28.

It should be carefully noted that

$$\int_{-\infty}^{\infty} f(x)\,dx \tag{10.10.29}$$

is an improper integral. Technically, this integral is defined as the following sum of limits, both of which must exist:

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_0^R f(x)\,dx + \lim_{S\to\infty}\int_{-S}^{0} f(x)\,dx \tag{10.10.30}$$

When we use the residue theorem we are in fact computing

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx \tag{10.10.31}$$

which may exist even though the limits 10.10.30 do not exist.[13] We call the value of the limit in 10.10.31 the *Cauchy principle value* of

$$\int_{-\infty}^{\infty} f(x)\,dx$$

Of course, if the two limits in Eq. 10.10.30 exist, then the principal value exists and is the same limit.

[13]Note $\lim\limits_{R\to\infty}\int_{-R}^{R} x\,dx = 0$ but neither $\lim\limits_{R\to\infty}\int_0^R x\,dx$ nor $\lim\limits_{s\to\infty}\int_{-s}^{0} x\,dx$ exist.

EXAMPLE 10.10.1

Find the value of the following integrals, where C is the circle $|z| = 2$.

(a) $\displaystyle\oint_C \frac{\cos z}{z^3}\,dz$ (b) $\displaystyle\oint_C \frac{dz}{z^2+1}$ (c) $\displaystyle\oint_C \frac{z^2-2}{z(z-1)(z+4)}\,dz$ (d) $\displaystyle\oint_C \frac{z}{(z-1)^3(z+3)}\,dz$

▶ Solution

(a) We expand the function $\cos z$ as

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

EXAMPLE 10.10.1 *(Continued)*

The integrand is then

$$\frac{\cos z}{z^3} = \frac{1}{z^3} - \frac{1}{2z} + \frac{z}{4!} + \cdots$$

The residue, the coefficient of the $1/z$ term, is

$$b_1 = -\frac{1}{2}$$

Thus, the value of the integral is

$$\oint_C \frac{\cos z}{z}\, dz = 2\pi i \left(-\frac{1}{2}\right) = -\pi i$$

(b) The integrand is factored as

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

Two singularities exist inside the circle of interest. The residue at each singularity is found to be

$$(b_1)_{z=i} = (z-i)\frac{1}{(z+i)(z-i)}\bigg|_{z=i} = \frac{1}{2i}$$

$$(b_1)_{z=-i} = (z+i)\frac{1}{(z+i)(z-i)}\bigg|_{z=-i} = -\frac{1}{2i}$$

The value of the integral is

$$\oint_C \frac{dz}{z^2+1} = 2\pi i \left(\frac{1}{2i} - \frac{1}{2i}\right) = 0$$

Moreover, this is the value of the integral around every curve that encloses the two poles.
(c) There are two poles of order 1 in the region of interest, one at $z = 0$ and the other at $z = 1$. The residue at each of these poles is

$$(b_1)_{z=0} = z\frac{z^2-2}{z(z-1)(z+4)}\bigg|_{z=0} = \frac{1}{2}$$

$$(b_1)_{z=1} = (z-1)\frac{z^2-2}{z(z-1)(z+4)}\bigg|_{z=1} = -\frac{1}{5}$$

The integral is

$$\oint_C \frac{z^2-2}{z(z-1)(z+4)}\, dz = 2\pi i \left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3\pi i}{5}$$

EXAMPLE 10.10.1 *(Continued)*

(d) There is one pole in the circle $|z| = 2$, a pole of order 3. The residue at that pole is (see Eq. 10.10.10)

$$b_1 = \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-1)^3\frac{z}{(z-1)^3(z+3)}\right]_{z=1}$$
$$= \frac{1}{2}\frac{-6}{(z+3)^3}\bigg|_{z=1} = -\frac{3}{64}$$

The value of the integral is then

$$\oint_C \frac{z}{(z-1)^3(z+3)}\,dz = 2\pi i\left(-\frac{3}{64}\right) = -0.2945i$$

EXAMPLE 10.10.2

Evaluate the real integral

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$$

▶ Solution

Using Eqs. 10.10.13, the integral is transformed as follows:

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \oint_C \frac{dz/iz}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)} = -2i\oint \frac{dz}{z^2+4z+1}$$

where C is the unit circle. The roots of the denominator are found to be

$$z = -2 \pm \sqrt{3}$$

Hence, there is a zero at $z = -0.2679$ and at $z = -3.732$. The first of these zeros is located in the unit circle, so we must determine the residue at that zero; the second is outside the unit circle, so we ignore it. To find the residue, write the integrand as partial fractions

$$\frac{1}{z^2+4z+1} = \frac{1}{(z+0.2679)(z+3.732)} = \frac{0.2887}{z+0.2679} + \frac{-0.2887}{z+3.732}$$

The residue at the singularity in the unit circle is then the coefficient of the $(z+0.2679)^{-1}$ term. It is 0.2887. Thus, the value of the integral is, using the residue theorem,

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = -2i(2\pi i \times 0.2887) = 3.628$$

EXAMPLE 10.10.3

Evaluate the real integral

$$\int_0^{\infty} \frac{dx}{(1+x^2)}$$

Note that the lower limit is zero.

► Solution

We consider the complex function $f(z) = 1/(1+z^2)$. Two poles exist at the points where

$$1 + z^2 = 0$$

They are

$$z_1 = i \quad \text{and} \quad z_2 = -i$$

The first of these roots lies in the upper half-plane. The residue there is

$$(b_1)_{z=i} = (z-i)\frac{1}{(z-i)(z+i)}\bigg|_{z=i} = \frac{1}{2i}$$

The value of the integral is then (refer to Eq. 10.10.23)

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i\left(\frac{1}{2i}\right) = \pi$$

Since the integrand is an even function,

$$\int_0^{\infty} \frac{dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

Hence,

$$\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$$

EXAMPLE 10.10.4

Determine the value of the real integrals

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx$$

► Solution

To evaluate the given integrals refer to Eqs. 10.10.25 through 10.10.28. Here

$$I_1 = \oint_C \frac{e^{iz}}{1+z^2}\,dz$$

EXAMPLE 10.10.4 (*Continued*)

and C is the semicircle in Fig. 10.20. The quantity $(1+z^2)$ has zeros as $z = \pm i$. One of these points is in the upper half-plane. The residue at $z = i$ is

$$(b_1)_{z=i} = (z-i)\frac{e^{iz}}{1+z^2}\bigg|_{z=i} = \frac{e^{-1}}{2i} = -0.1839i$$

The value of the integral is then

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = 2\pi i(-0.1839i) = 1.188$$

The integral can be rewritten as

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx + i\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx$$

Equating real and imaginary parts, we have

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx = 1.188 \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx = 0$$

The result with $\sin x$ is not surprising since the integrand is an odd function, and hence

$$\int_{0}^{\infty} \frac{\sin x}{1+x^2}\,dx = -\int_{-\infty}^{0} \frac{\sin x}{1+x^2}\,dx$$

10.10.1 *Maple* Applications

The partial fraction decomposition of Example 10.10.2 can be performed by *Maple,* provided that we use the `real` option of the `convert/parfrac` command:

```
>convert(1/(z^2+4*z+1), parfrac, z, real);
```

$$-\frac{0.2886751345}{z+3.732050808} + \frac{0.2886751345}{z+0.2679491924}$$

Note that *Maple* will compute the real integral of this problem, but it is not clear what method is being used:

```
>int(1/(2+cos(theta)), theta=0..2*Pi);
```

$$\frac{2\pi\sqrt{3}}{3}$$

```
>evalf(%);
```

$$3.627598730$$

Problems

Find the residue of each function at each pole.

1. $\dfrac{1}{z^2+4}$

2. $\dfrac{z}{z^2+4}$

3. $\dfrac{1}{z^2}\sin 2z$

4. $\dfrac{e^z}{(z-1)^2}$

5. $\dfrac{\cos z}{z^2+2z+1}$

6. $\dfrac{z^2+1}{z^2+3z+2}$

Evaluate each integral around the circle $|z|=2$.

7. $\displaystyle\oint \frac{e^z}{z^4}\,dz$

8. $\displaystyle\oint \frac{\sin z}{z^3}\,dz$

9. $\displaystyle\oint \frac{z^2}{1-z}\,dz$

10. $\displaystyle\oint \frac{z+1}{z+i}\,dz$

11. $\displaystyle\oint \frac{z\,dz}{z^2+4z+3}$

12. $\dfrac{dz}{4z^2+9}$

13. $\displaystyle\oint \frac{e^{1/z}}{z}\,dz$

14. $\displaystyle\oint \frac{\sin z}{z^3-z^2}\,dz$

15. $\displaystyle\oint e^z \tan z\,dz$

16. $\displaystyle\oint \frac{z^2+1}{z(z+1)^3}\,dz$

17. $\displaystyle\oint \frac{\sinh\,\pi z}{z^2+1}\,dz$

18. $\displaystyle\oint \frac{\cosh \pi z}{z^2+z}\,dz$

Determine the value of each real integral.

19. $\displaystyle\int_0^{2\pi} \frac{\sin\theta}{1+\cos\theta}\,d\theta$

20. $\displaystyle\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$

21. $\displaystyle\int_0^{2\pi} \frac{d\theta}{5-4\cos\theta}$

22. $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+2\sin\theta}$

23. $\displaystyle\int_0^{2\pi} \frac{\sin 2\theta\,d\theta}{5+4\cos\theta}$

24. $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta\,d\theta}{5-4\cos\theta}$

Evaluate each integral.

25. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$

26. $\displaystyle\int_0^{\infty} \frac{x^2\,dx}{(1+x^2)^2}$

27. $\displaystyle\int_{-\infty}^{\infty} \frac{1+x}{1+x^3}\,dx$

28. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{x^4+x^2+1}$

29. $\displaystyle\int_0^{\infty} \frac{x^2\,dx}{1+x^6}$

30. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+5x^2+2}$

31. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 2x}{1+x}\,dx$

32. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^2}\,dx$

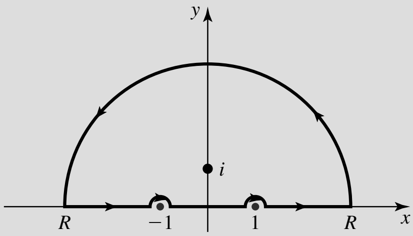
33. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{1+x^2}\,dx$

34. $\displaystyle\int_0^{\infty} \frac{\cos x}{1+x^4}\,dx$

35. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+3x+2}\,dx$

36. $\displaystyle\int_0^\infty \frac{\cos 4x}{(1+x^2)^2}\,dx$

37. Find the value of $\int_{-\infty}^{\infty} dx/(x^4-1)$ following the technique using the path of integration of Fig. 10.20, but integrate around the two poles on the x axis by considering the path of integration shown.



38. Prove that the exact value of the first integral in Example 10.10.4 is $\pi(\cosh(1)-\sinh(1))$.

Use *Maple* to evaluate:

39. Problem 19

40. Problem 20

41. Problem 21

42. Problem 22

43. Problem 23

44. Problem 24

45. Problem 25

46. Problem 26

47. Problem 27

48. Problem 28

49. Problem 29

50. Problem 30

51. Problem 31

52. Problem 32

53. Problem 33

54. Problem 34

55. Problem 35

56. Problem 36